IGUSA STACKS AND THE COHOMOLOGY OF SHIMURA VARIETIES II

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Abstract. We construct Igusa stacks for all Shimura varieties of abelian type and derive consequences for the cohomology of these Shimura varieties. As an application, we prove that the Fargues–Scholze local Langlands correspondence agrees with the semi-simplification of the local Langlands correspondences constructed by Arthur, Mok and others, for all classical groups of type A, B and D; this extends work of Hamann, Bertoloni Meli–Hamann–Nguyen and Peng.

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1. Introduction

1.1. **Overview.** Igusa stacks are geometric objects, akin to Shimura varieties, whose existence has been conjectured by Scholze, motivated by questions related to local-global compatibility in the Langlands program, see [Zha23, Conjecture 1.1]. The program to construct Igusa stacks was initiated in [Zha23] and continued in [DvHKZ24a] and [Kim25], and their existence has led to several breakthroughs in the study of the cohomology of Shimura varieties. In particular, advances have been made in the study of the generic part of the cohomology with torsion coefficients, see [HL23] and [YZ25], the generalized Ihara's lemma for definite unitary groups has been established, see [Yan25], and a very general Eichler–Shimura relation has been proven for Shimura varieties of Hodge type, see [vdH24].

In this paper, we construct Igusa stacks for all Shimura varieties of abelian type, and we use these to prove new results about the cohomology of Shimura varieties. Our primary application is to the local Langlands correspondence for classical groups of types A, B, D, as well as GSp_4, over arbitrary p-adic local fields. For these groups, we show that the Fargues–Scholze local Langlands correspondence agrees with the semisimplification of the local Langlands correspondences of Arthur, Mok and others, see [Art13], [Mok15], [CZ21], [KMSW14], [Ish24], [GT11a], [GT14]. Our work extends [Ham25], [BMHN24], [Pen25b], where the analogous result is established for unramified groups over unramified extensions of $\mathbb{Q}_p$ with $p > 2$.

In addition, we prove the Eichler–Shimura relation of Blasius–Rogawski [BR94], at Iwahori level and with torsion coefficients, generalizing [vdH24]. We furthermore show that the local plectic conjecture of Nekovár–Scholl holds at split primes, extending results of [Lee22] and [LH22]; the general case is being studied by Feng–Tamiozzo–Zhang.

1.2. **Main geometric results.** Let (G, X) be a Shimura datum of abelian type with reflex field E. Let p be a prime number, let v be a prime of E above p and set $E = \mathsf{E}_v$ and $G = \mathsf{G} \otimes \mathbb{Q}_p$. Let $K \subset \mathsf{G}(\mathbb{A}_f)$ be a neat compact open subgroup and consider the Shimura variety $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$ of level K over E. There is an open subspace (the "good reduction locus")

$$\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\circ,\mathrm{an}} \subset \mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\mathrm{an}}$$

of the adic space $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{\mathrm{an}}$ associated with $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$, which is compatible with changing the level K. Let Perf denote the category of perfectoid spaces of characteristic p. We let $\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{\circ,\diamondsuit} \subset \mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{\diamondsuit}$ be the corresponding objects with infinite level at p, considered as v-sheaves on Perf. These come equipped with a Hodge–Tate period map

$$\pi_{\mathrm{HT}}^{\diamondsuit} \colon \mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{\diamondsuit} \to \mathrm{Gr}_{G,\mu^{-1}},$$

where $\mathrm{Gr}_{G,\mu^{-1}}$ is the Schubert cell in the $\mathbf{B}_{\mathrm{dR}}^+$-affine Grassmannian corresponding to the inverse of the Hodge cocharacter μ. Let Bun_G denote the stack of G-bundles on the Fargues–Fontaine curve. Finally we need the Beauville–Laszlo map $\mathrm{BL}: \mathrm{Gr}_{G,\mu^{-1}} \to \mathrm{Bun}_G$, whose image is an open substack $\mathrm{Bun}_{G,\mu^{-1}}$ of Bun_G.

Theorem I (Theorem 4.4.1, 4.5.1)**.** *There is an open immersion of v-stacks* $\mathrm{Igs}^{\circ}_{K^p}(\mathsf{G}, \mathsf{X}) \hookrightarrow \mathrm{Igs}_{K^p}(\mathsf{G}, \mathsf{X})$ *on* Perf *sitting in a commutative diagram*

$$\begin{tikzcd}
\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{\circ,\diamondsuit} \arrow[r, hook] \arrow[d, "q^{\circ}_{\mathrm{Igs}}"] & \mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{\diamondsuit} \arrow[r, "\pi_{\mathrm{HT}}^{\diamondsuit}"] \arrow[d, "q_{\mathrm{Igs}}"] & \mathrm{Gr}_{G,\mu^{-1}} \arrow[d, "\mathrm{BL}"] \\
\mathrm{Igs}^{\circ}_{K^p}(\mathsf{G}, \mathsf{X}) \arrow[r, hook] & \mathrm{Igs}_{K^p}(\mathsf{G}, \mathsf{X}) \arrow[r, "\overline{\pi}_{\mathrm{HT}}"] & \mathrm{Bun}_{G,\mu^{-1}},
\end{tikzcd}$$

with both squares Cartesian. Moreover, for $\ell \neq p$, both $\mathrm{Igs}^{\circ}_{K^p}(\mathsf{G}, \mathsf{X})$ *and* $\mathrm{Igs}_{K^p}(\mathsf{G}, \mathsf{X})$ *are ℓ-cohomologically smooth of dimension zero with constant dualizing sheaf. Furthermore,* $\varprojlim_{K^p} \mathrm{Igs}^{\circ}_{K^p}(\mathsf{G}, \mathsf{X})$ *and* $\varprojlim_{K^p} \mathrm{Igs}_{K^p}(\mathsf{G}, \mathsf{X})$ *satisfy* [Kim25, Definition 5.4]*.*

This confirms a conjecture of Scholze, see [Zha23, Conjecture 1.1.(4)]. For PEL-type Shimura varieties at unramified primes p, the existence of $\mathrm{Igs}^{\circ}(\mathsf{G}, \mathsf{X})$ was proved by one of us (MZ) as part of [Zha23, Theorem 1.3]. For Shimura varieties of Hodge type, it is [DvHKZ24a, Theorem I] or alternatively [Kim25, Theorem D]. For certain abelian type Shimura varieties closely related to the PEL cases of [Zha23, Theorem 1.3], it is [Sch26a, Theorem 1.3]. The existence of $\mathrm{Igs}(\mathsf{G}, \mathsf{X})$ for Shimura varieties of Hodge type was proved in [Kim25, Theorem D]. Note that the Igusa stacks of Theorem 4.4.1 are functorial in morphisms of Shimura data by [Kim25, Theorem B], and in particular unique. We remark also that in [LH26], Li-Huerta studies the global function field analog of the Igusa stack and establishes the analogous fiber product diagram, see [LH26, Theorem D].

We will discuss the proof of Theorem I in Section 1.5. First, we will discuss applications to the local Langlands correspondence for classical groups, and to the cohomology of Shimura varieties.

1.3. **Compatibility of local Langlands correspondences.** Let F be a p-adic local field and let G be a connected reductive group over F. The (conjectural) local Langlands correspondence is a map

$$\mathrm{LL}_G : \Pi_G \to \Phi(G),$$

where Π_G is the set of isomorphism classes of irreducible smooth $\mathbb{C}$-linear representations of $G(F)$, and $\Phi(G)$ is the set of $\hat{G}$-conjugacy classes of L-parameters $W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG(\mathbb{C})$, see Section 7.1.1. Fargues and Scholze [FS24] have recently constructed a map[1]

$$\mathrm{LL}_G^{\mathrm{FS}} : \Pi_G \to \Phi^{\mathrm{ss}}(G),$$

where the target is now the set of $\hat{G}$-conjugacy classes of semisimple L-parameters $W_F \to {}^LG(\mathbb{C})$. This map is expected to recover the composition of (the conjectural) map LL_G with the semisimplification map $(-)^{\mathrm{ss}} : \Phi(G) \to \Phi^{\mathrm{ss}}(G)$, see Section 7.1.6.

1.3.1. By work of Arthur [Art13], Mok [Mok15], and many others[2] a map LL_G has been constructed for many classical groups. More precisely, we consider the following cases.

- U The group G is an inner form of a quasi-split unitary group over F, and the map LL_G is the one constructed by Mok [Mok15, Theorem 2.5.1, 3.2.1] if G is quasi-split and by Kaletha–Minguez–Shin–White [KMSW14, Theorem 1.6.1] if G is not quasi-split.
- B The group G is an inner form of the split special orthogonal group $\mathrm{SO}_{2n+1,F}$, and the map LL_G is the one constructed by Arthur [Art13, Theorem 1.5.1] if G is quasi-split and by Ishimoto [Ish24, Theorem 1.2] if G is not quasi-split.
- C_2 The group G is an inner form of GSp_4, and the map LL_G is the one constructed by Gan–Takeda [GT11a, Main theorem] if G is split and Gan–Tantono [GT14, Main Theorem] if G is nonsplit.
- D The group G is a pure inner form of a quasi-split special orthogonal group $\mathrm{SO}(V')$ for an even dimensional quadratic space V' over F (which implies that $G \simeq \mathrm{SO}(V)$ for some quadratic space V over F of the same dimension as V').

[1] This map depends a priori on a choice of ℓ, a choice of $\sqrt{p} \in \overline{\mathbb{Q}}_\ell$ and a choice of isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. By recent work of Scholze [Sch25, Theorem 1.1], it is independent of these choices if we choose ι to take the fixed $\sqrt{p}$ to the positive square root of p in $\mathbb{R}$.

[2] The work of Arthur and hence the subsequent work of Mok and many others is conditional on several preprints of Arthur which have not yet appeared. By recent work of Atobe–Gan–Ichino–Kaletha–Minguez–Shin [AGI+24], the work of Arthur is now only conditional on the twisted weighted fundamental lemma, see [Shi24, Assumption H1].

Noting that the dual group of $\hat{G}$ is SO_{2n}, which has a conjugation action by O_{2n}, we consider the map

$$\widetilde{\mathrm{LL}}_G : \Pi_G \to \widetilde{\Phi}(G) = \Phi(G)/\,\mathrm{O}_{2n},$$

constructed by Arthur [Art13, Theorem 1.5.1] if G is quasi-split and by Chen–Zou [CZ21, Theorem A.2] if G is not quasi-split. In this case, we define $\widetilde{\mathrm{LL}}_G^{\mathrm{FS}}$ as the composition of $\mathrm{LL}_G^{\mathrm{FS}}$ with the map $\Phi^{\mathrm{ss}}(G) \to \Phi^{\mathrm{ss}}(G)/\,\mathrm{O}_{2n}$.

Theorem II. *In cases U, B, C_2 listed in Section 1.3.1, we have the equality $(-)^{\mathrm{ss}} \circ \mathrm{LL}_G = \mathrm{LL}_G^{\mathrm{FS}}$. In case D, we have the equality $(-)^{\mathrm{ss}} \circ \widetilde{\mathrm{LL}}_G = \widetilde{\mathrm{LL}}_G^{\mathrm{FS}}$.*

Remark 1.3.2. If F is a finite unramified extension of $\mathbb{Q}_p$ for $p > 2$ and G is unramified over F, then Theorem II was previously known by work of Hamann [Ham25] (case C_2), and Peng [Pen25b] (cases U, B, D). If moreover $F = \mathbb{Q}_p$ and G is a unitary group in an odd number of variables, then case U was dealt with in earlier work of Bertoloni Meli–Hamann–Nguyen [BMHN24]. A similar theorem for GL_N and its inner forms is proved in earlier work of Fargues–Scholze [FS24] and Hansen–Kaletha–Weinstein [HKW22].

Corollary III (Theorem 7.4.4)**.** *Let F be a p-adic local field and let G be a pure inner form of an even orthogonal group over F. There is a unique local Langlands correspondence $\mathrm{LL}_{\mathrm{SO}_{2n}} : \Pi_{\mathrm{temp}} \to \Phi_{\mathrm{temp}}(G)$ lifting $\widetilde{\mathrm{LL}}_{\mathrm{SO}_{2n}}$ such that $(-)^{\mathrm{ss}} \circ \mathrm{LL}_{\mathrm{SO}_{2n}} = \mathrm{LL}_{\mathrm{SO}_{2n}}^{\mathrm{FS}}$. Moreover, it satisfies the conditions listed in* [Pen25b, Theorem 7.1.1].

Corollary III is proved in [Pen25b] if $p > 2$ and F is unramified, see [Pen25b, Theorem 7.1.1]. Given Theorem II, his proof goes through verbatim.

Remark 1.3.3. As observed by Peng and Hamann, the strategy of [Ham25], [BMHN24], and [Pen25b] would work essentially verbatim for groups over ramified local fields, if basic uniformization were established for certain Shimura varieties associated to Shimura data (G, X) of abelian type at primes p where G is ramified. Nevertheless, our focus in this work is on the construction of the Igusa stack and its applications, so we do not study basic uniformization in this article, and our proofs do not rely upon it.

Moreover, proving basic uniformization in full generality does not seem straightforward. With some work, the abelian type case likely reduces to the case of suitably chosen Hodge type Shimura varieties, but already in these Hodge type cases, the result is not known. To generalize the strategy of [HZZ25, Proposition 5.2.2], the missing ingredient is a CM lifting result for (basic) mod p isogeny classes. For quasi-split groups this is [KZ25, Theorem 2.2.7] under an assumption at $p = 2$, but this assumption notably excludes ramified unitary groups. The authors have been informed that, in forthcoming work [SW26], Xu Shen and Peihang Wu will prove basic uniformization for Shimura varieties of abelian type under the restriction $p \neq 2$.

1.3.4. We now explain the proof of Theorem II in the quasi-split case; for notational simplicity, we focus on the case of odd orthogonal groups. We first reduce to the case of supercuspidal representations π using induction on the rank of our classical group and compatibility of semisimplified L-parameters with parabolic induction; this part of our argument closely follows [Ham25, BMHN24, Pen25b]. The case of supercuspidal π moreover breaks up into two subcases depending on whether $\mathrm{LL}_G(\pi)$ is a supercuspidal L-parameter or not.

If $\phi = \mathrm{LL}_G(\pi)$ is not supercuspidal, then the L-packet of ϕ will contain a parabolically induced representation π' of $G(\mathbb{Q}_p)$. By the induction hypothesis, we know that $(-)^{\mathrm{ss}} \circ \mathrm{LL}_G(\pi') = \mathrm{LL}_G^{\mathrm{FS}}(\pi')$, and we would like to deduce the same statement for the other elements of the L-packet. For this, we can apply recent work of Hansen [Han26, Theorem 1.1] and Varma [Var24, Theorem 4.4.2], to deduce the constancy on the L-packet of the Fargues–Scholze L-parameter, see [Han26, Corollary 1.2].

If $\phi = \mathrm{LL}_G(\pi)$ is supercuspidal, then all the members π of the L-packet of ϕ will be cuspidal, and we give a direct proof by a globalization argument: First we choose a totally real field $\mathsf{L}^+ \neq \mathbb{Q}$ with a finite place v of L^+ such that $\mathsf{L}^+_v = F$. We then globalize G to a connected reductive group H over L^+ which is compact at all but one infinite place of L^+, so that $\mathsf{G} = \mathrm{Res}_{\mathsf{L}^+/\mathbb{Q}}$ admits a Shimura datum X of abelian type.[3] We then globalize π to a cohomological cuspidal automorphic representation Π of $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+}) = \mathsf{G}(\mathbb{A})$ with good properties (e.g. cohomological of regular weight, isomorphic to a twist of Steinberg at an auxiliary place). It now follows from work of Arthur [Art13], Kisin–Shin–Zhu [KSZ21] and others that we understand the Π^∞-isotypic part of the cohomology of the Shimura varieties for (G, X). More precisely, write

$$r_\mu : {}^L H \to \mathrm{GL}(V_\mu)$$

for the representation of the L-group of H associated to X and v. For example, in the case of odd orthogonal groups SO_{2n+1} under consideration, this is just the standard representation

$$\mathrm{Sp}_{2n} \xhookrightarrow{\mathrm{Std}} \mathrm{GL}_{2n}\,.$$

Then by work of Arthur, Mok and others (see Theorem 6.3.5) there is an irreducible global Galois representation $R_\Pi : \mathrm{Gal}_{\mathsf{L}^+} \to \mathrm{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ such that (up to Tate-twist)

$$R_\Pi\big|_{W_F} \simeq r_\mu \circ \phi.$$

The complex $R\Gamma(\mathbf{Sh}(\mathsf{G}, \mathsf{X}), \mathbb{W})[\Pi^\infty]$, where $\mathbb{W}$ is an automorphic $\overline{\mathbb{Q}}_\ell$ local system corresponding to Π_∞, is concentrated in middle degree. By work of Kisin–Shin–Zhu [KSZ21], its middle cohomology group is isomorphic up to semisimplification to $R_\Pi^{\oplus m}$ for some $m \in \mathbb{Z}_{\geq 0}$ (up to the same Tate-twist). It is at this point that we use the existence of Igusa stacks through Corollary 6.4.2, to deduce that

$$\left(R_\Pi\big|_{W_F}\right)^{\mathrm{ss}} \simeq r_\mu \circ \mathrm{LL}_G^{\mathrm{FS}}(\pi). \tag{1.3.1}$$

To deduce this from Corollary 6.4.2, we crucially use the supercuspidality of $\phi = \mathrm{LL}_G(\pi)$ which implies that all the irreducible W_E-subquotients of $R_\Pi^{\oplus m}\Big|_{W_F}$ occur

[3] The Shimura datum (G, X) is essentially never of Hodge type.

with multiplicity 1. To conclude, we note that the equality eq. (1.3.1) together with work of Gan–Gross–Prasad [GGP12, Theorem 8.1] implies that the L-parameters ϕ and $\mathrm{LL}_G^{\mathrm{FS}}(\pi)$ are conjugate to one another.

1.3.5. The above argument does not work for non quasi-split groups, because it is not necessarily true that if ϕ is not supercuspidal, then its L-packet must contain a parabolically induced representation. Instead, we show that the conclusion of Theorem II propagates from quasi-split groups G^* to their extended pure inner forms $G = G_b$. We deduce this from another recent result of Hansen [Han26, Theorem 1.4] together with the endoscopic character relations between G and G^* for LL_G and LL_{G^*}.

Remark 1.3.6. It is crucial that we can take r_μ to be the standard representation of Sp_{2n}, since otherwise $\mathrm{LL}_G^{\mathrm{FS}}(\pi)$ cannot be recovered from $r_\mu \circ \mathrm{LL}_G^{\mathrm{FS}}(\pi)$. This explains why we cannot prove Theorem II for GSp_{2n} with $n \geq 3$: The representation r_μ of the dual group GSpin_{2n+1} coming from the Hodge cocharacter μ of a Shimura datum for GSp_{2n} is given by the 2^n-dimensional spin representation. For GSp_{2n}, we do prove a statement after composition with the spin representation, under some assumptions, see Corollary 6.4.2.

1.4. **Cohomological results.** Let (G, X) be a Shimura datum of abelian type with reflex field E and suppose that $\mathsf{G} = \mathrm{Res}_{\mathsf{F}/\mathbb{Q}} \mathsf{H}$ for a totally real field F and a connected reductive group H over F; let $K \subset \mathsf{G}(\mathbb{A}_f)$ be a neat compact open subgroup. Nekovár–Scholl [NS16, Conjecture 6.1] predict that the action of $\mathrm{Gal}_{\mathsf{E}}$ on $R\Gamma_{\text{ét}}(\mathbf{Sh}_L(\mathsf{G}, \mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{Z}_\ell)$ extends to the action of a larger group, the plectic reflex Galois group. In particular, for a place v of E with completion E, the action of Gal_E should extend to the action of a local version of the plectic reflex Galois group.

Suppose that v lies above a prime p that splits completely in F. Then the local plectic reflex Galois group is a product $\prod_i \mathrm{Gal}_{E_i}$, see Section 5.7.2, where the product is indexed by places of F above p and each E_i is a finite extension of $\mathbb{Q}_p$. There is moreover a natural map $\mathrm{Gal}_E \to \prod_i \mathrm{Gal}_{E_i}$. Let ℓ be a prime distinct from p and let Λ be a $\mathbb{Z}/\ell^n\mathbb{Z}$-algebra containing a fixed square root of p.

Theorem IV (Theorem 5.7.3)**.** *Suppose that ℓ is coprime to the order of $\pi_0(Z(G))$. If $K = K_p K^p$ with K^p sufficiently small* (*see Definition 5.3.3*)*, then the cohomology complex*

$$R\Gamma(\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})_{\overline{E}}, \Lambda) \in D(W_E, \Lambda)$$

lifts canonically to an object in $D(\prod_i W_{E_i}, \Lambda)$, compatibly with the Hecke actions.

Remark 1.4.1. This is proved at unramified primes p and with $\overline{\mathbb{Z}}_\ell$ coefficients for the basic part of the cohomology of the Shimura variety by Li-Huerta [LH22, Theorem A].

If K_p is hyperspecial, then the action of Gal_E on the cohomology of the Shimura variety is unramified. Theorem IV then gives us commuting partial Frobenius operators acting on the cohomology. It turns out these partial Frobenii satisfy a version of the Eichler–Shimura relation of Blasius–Rogawski [BR94, Section 6].

Corollary V (Corollary 5.7.6)**.** *Suppose that ℓ is coprime to the order of $\pi_0(Z(G))$. If $K = K_pK^p$ with K_p hyperspecial and with K^p sufficiently small, then there exist mutually commuting partial Frobenius operators σ_i acting on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}}, \Lambda)$, such that for any lift σ of Frob_q, we have*

$$\sigma = \prod_{i=1}^{r} \sigma_i$$

as endomorphisms of $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}}, \Lambda)$. Moreover, each σ_i is annihilated by the renormalized Hecke polynomial $H_{\mu_i}(X) \in \mathcal{H}_{\mathcal{G}_i}[X]$ constructed in Section 5.6.5.

Note that Corollary V has previously been announced by Zhu [Zhu25a, Theorem 3.2.2]. When $E = \mathbb{Q}_p$, Corollary 5.7.6 is also proved by Lee [Lee22, Theorem 1.0.4] by a completely different method.

Remark 1.4.2. Corollary V is new even when $\mathsf{F} = \mathbb{Q}$ and $E \neq \mathbb{Q}_p$. In this case, it verifies the conjectural Eichler–Shimura relation of Blasius–Rogawski at hyperspecial level, see [BR94, Section 6]. See [Wu21, Corollary 1.2] for the Hodge type case, and [DvHKZ24a, the discussion preceding Theorem III] for an overview of previous work on the conjecture.

Remark 1.4.3. We also prove a version of Corollary V at Iwahori level, see Theorem 5.6.6. This generalizes [vdH24, Theorem 1.1], which deals with the case of Hodge type Shimura varieties, generalizing [DvHKZ24a, Theorem III] which dealt only with the case that G is unramified. Our proof uses the methods of van den Hove [vdH24] in combination with Theorem 5.3.11.

1.4.4. We now give a brief summary of our other cohomological results. For the sake of brevity, and because many are generalizations of results proved in [DvHKZ24a], we do not state these results in full here.

Theorem 5.3.11 expresses the (compactly supported) cohomology of the Shimura variety of level K^p in terms of a Hecke operator applied to the sheaf on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$, given by the (exceptional) pushforward of a constant sheaf along $\mathrm{Igs}_{K^p}(\mathsf{G},\mathsf{X})_{\overline{\mathbb{F}}_p} \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$. The Hodge type analogue is [DvHKZ24a, Theorem IV], although we give a different proof.

Theorem 5.5.10 gives a weak compatibility result between the cohomology of Shimura varieties and the Fargues–Scholze local Langlands correspondence, it is the abelian type analogue of [DvHKZ24a, Theorem II] and is proved analogously. For compact Shimura varieties of abelian type, we use Matsushima's formula to prove a strengthening of Theorem 5.5.10, see Corollary 6.1.2. This corollary is crucially used in the proof of Theorem II.

Theorem 5.4.2 is a version of Mantovan's product formula (see [Man04, Theorem 1]) expressing the compactly supported cohomology of the Shimura varieties in terms of the compactly supported cohomology of the fibers of π_{HT}[4] and the cohomology of

[4]The Mantovan product formula is usually stated in terms of the cohomology of the fibers of $\pi_{\mathrm{HT}}^{\circ}$, which can often be identified with the cohomology of perfect Igusa varieties.

local Shimura varieties. In the non-compact case, this generalizes [HL23, Proposition 3.17] and in the compact case it generalizes [DvHKZ24a, Theorem 8.5.7].

Remark 1.4.5. We do not prove a generalization of [DvHKZ24a, Theorem VII], which gives fiber product diagrams on the level of canonical integral models of Shimura varieties. However, see Conjecture 2.3.1 for a precise conjecture.

Remark 1.4.6. We do not address torsion vanishing in this paper, since the results of [YZ25] apply to most Shimura varieties of abelian type with parahoric level at p. For the same reason, we do not attempt to prove perversity of the Igusa sheaf, but see Conjecture 5.2.4 for a precise conjecture.

Remark 1.4.7. We expect a version of Theorem 5.3.11 to hold with coefficients in Berkovich motives as in [Sch26b] and [Sch25]. Taking ℓ-adic realizations for $\ell \neq p$ this should then recover Theorem 5.3.11 as well as a version with $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$-coefficients. More interestingly, there is a p-adic or Hyodo–Kato realization (as in [ABLB$^+$25]) of Berkovich motives, see [Aok26, Theorem 1.7]. The Hyodo–Kato realization of the Igusa sheaf is expected to encode the rigid cohomology of Igusa varieties, see [ABLB$^+$25, Remark 1.4.7].

Remark 1.4.8. Recently, Scholze has suggested in a talk [Sch24, 49:00 onwards] that there should be a "locally analytic" version of the Igusa stack diagram which might be used to prove results about the locally analytic vectors in the completed cohomology of Shimura varieties; this is the subject of a work-in-progress of Anschütz–Le Bras–Rodríguez Camargo–Scholze. A variant of such a locally analytic Igusa stack diagram is worked out in [How26, Section 10], using the existence of an Igusa stack as an input.

1.5. **The proof of Theorem I.** Let $(\mathsf{G}_2, \mathsf{X}_2)$ be a Shimura datum of abelian type with reflex field E_2. Work of Deligne and Lovering, [Del79], [Lov17b], tells us that we can find a diagram of Shimura data

$$\begin{tikzcd} & (\mathsf{G}_3, \mathsf{X}_3) \arrow[dl] \arrow[dr] & \\ (\mathsf{G}_2, \mathsf{X}_2) & & (\mathsf{G}, \mathsf{X}), \end{tikzcd}$$

where (G, X) is of Hodge type, where the left arrow is an ad-isomorphism and where the right arrow induces an isomorphism of derived groups. We moreover have fairly precise control over the reflex fields of (G, X) and $(\mathsf{G}_3, \mathsf{X}_3)$. We will start with the Igusa stack $\mathrm{Igs}(\mathsf{G}, \mathsf{X}) := \varprojlim_{K^p} \mathrm{Igs}_{K^p}(\mathsf{G}, \mathsf{X})$ and construct an Igusa stack for $(\mathsf{G}_2, \mathsf{X}_2)$, the $\circ$-variants are treated in the same way.

Step 1: We construct Igusa stacks for tori using the geometric interpretation of class field theory of Fargues and Fargues–Scholze. More precisely, we consider the v-sheaf $[\underline{\mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f)} / \underline{T(\mathbb{Q}_p)}]$ over $\overline{\mathbb{F}}_p$ and explicitly write down a Galois descent datum for $\overline{\mathbb{F}}_p/\mathbb{F}_p$ using global class field theory and the Hodge cocharacter μ. This defines the Igusa stack $\mathrm{Igs}(\mathsf{T}, \{\mu\})$ over $\mathbb{F}_p$. To check that this is an Igusa stack, we explicitly write down a presentation of $\mathrm{Bun}_{T,\mu^{-1}}$ over $\mathrm{Spd}\,\mathbb{F}_p$, and give an explicit formula for the Beauville–Laszlo map $\mathrm{BL}\colon \mathrm{Spd}\,E \to \mathrm{Bun}_T$ in this presentation. To do this, we reduce to the case that $T_{\mathbb{Q}_p} = \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$ for some finite extension E of $\mathbb{Q}_p$, and then use the geometric interpretation of Lubin–Tate theory (for E) of Fargues and Fargues–Scholze. Finally, we take the base change of $\mathrm{Igs}(\mathsf{T},\{\mu\})$ along BL and show that it recovers the canonical model of $\mathbf{Sh}(\mathsf{T},\{\mu\})$. This is carried out in Section 3.

Step 2: To construct Igusa stacks for $(\mathsf{G}_3,\mathsf{X}_3)$, we note that following diagram is Cartesian (where ab means maximal abelian quotient)

$$\begin{tikzcd} (\mathsf{G}_3,\mathsf{X}_3) \arrow[r] \arrow[d] & (\mathsf{G}_3^{\mathrm{ab}},\mathsf{X}_3^{\mathrm{ab}}) \arrow[d] \\ (\mathsf{G},\mathsf{X}) \arrow[r] & (\mathsf{G}^{\mathrm{ab}},\mathsf{X}^{\mathrm{ab}}). \end{tikzcd}$$

Thus $(\mathsf{G}_3,\mathsf{X}_3)$ sits inside $(\mathsf{G}_3^{\mathrm{ab}},\mathsf{X}_3^{\mathrm{ab}}) \times (\mathsf{G},\mathsf{X})$. Since (G,X) admits an Igusa stack by [Kim25, Theorem D], and $(\mathsf{G}_3^{\mathrm{ab}},\mathsf{X}_3^{\mathrm{ab}})$ admits an Igusa stack by Step 1, it is relatively straightforward to prove that their product admits an Igusa stack. We now apply [Kim25, Theorem C] to deduce that $(\mathsf{G}_2,\mathsf{X}_2)$ admits an Igusa stack.

Step 3: We descend the Igusa stack for $(\mathsf{G}_3,\mathsf{X}_3)$ down to an Igusa stack for $(\mathsf{G}_2,\mathsf{X}_2)$. The Shimura variety $\mathbf{Sh}(\mathsf{G}_3,\mathsf{X}_3)$ has an action of Deligne's locally profinite group $\mathcal{A}(\mathsf{G}_3)$, and it is well known[5] that $\mathbf{Sh}(\mathsf{G}_2,\mathsf{X}_2)$ is the pushout of $\mathbf{Sh}(\mathsf{G}_3,\mathsf{X}_3)$ along the natural morphism of locally profinite group schemes $\underline{\mathcal{A}(\mathsf{G}_3)} \to \underline{\mathcal{A}(\mathsf{G}_2)}$.

It follows from the functoriality results of one of us (DK) [Kim25, Theorem B] that $\underline{\mathcal{A}(\mathsf{G}_3)}$ acts on $\mathrm{Igs}(\mathsf{G}_3,\mathsf{X}_3)$, and in fact it acts through the quotient[6] $\underline{\mathcal{A}(\mathsf{G}_3)}^p := \underline{\mathcal{A}(\mathsf{G}_3)}/\underline{G_3(\mathbb{Q}_p)}$. Roughly speaking, we then show that the pushout of $\mathrm{Igs}(\mathsf{G}_3,\mathsf{X}_3)$ along $\underline{\mathcal{A}(\mathsf{G}_3)}^p \to \underline{\mathcal{A}(\mathsf{G}_2)}^p$ defines an Igusa stack for $(\mathsf{G}_2,\mathsf{X}_2)$. Our actual argument proceeds in a different way, using the perspective of uniformization maps of [Kim25], essentially because of subtleties of 2-categorical nature.

Step 4: We go from the Igusa stack $\mathrm{Igs}(\mathsf{G}_2,\mathsf{X}_2)$ to the finite level object $\mathrm{Igs}_{K^p}(\mathsf{G}_2,\mathsf{X}_2)$. This is straightforward if $Z_{\mathsf{G}}(\mathbb{Q}) \subset Z_{\mathsf{G}}(\mathbb{A}_f)$ is closed (i.e., Milne's axiom SV5 holds), where Z_{G} is the center of G. However, if SV5 fails then this is somewhat painful, see Corollary 4.5.4. Here again we use the formalism from [Kim25].

Step 5: We prove that the Igusa stack $\mathrm{Igs}_{K^p}(\mathsf{G}_2,\mathsf{X}_2)$ is ℓ-cohomologically smooth of ℓ-dimension zero, with trivial dualizing sheaf. If SV5 holds, then the proof of [DvHKZ24a, Theorem 8.3.1] essentially goes through. In general, we need to compute the relative dualizing sheaves of gerbes for locally profinite groups, which we do in Lemma 5.1.5.

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[5]We were not able to locate a proof in the literature, and so we have given a proof of this fact in Proposition 4.2.6 using work of Deligne [Del79].

[6]The group $G_3(\mathbb{Q}_p)$ injects into $\mathcal{A}(\mathsf{G}_3)$ but does not typically have closed image, thus our $\underline{\mathcal{A}(\mathsf{G}_3)}^p$ does not agree with the v-sheaf associated to the topological quotient $\mathcal{A}(G_3)/G_3(\mathbb{Q}_p)$.

1.6. **Organization of the paper.** Let us give a brief guide to help the reader navigate this paper.

Section 2 contains some preliminaries. In particular, we recall the abstract definition of Igusa stack given in [Kim25], and the equivalence between Igusa stacks and global uniformization maps, established in *loc. cit.*.

Sections 3 and 4 construct the Cartesian diagram in Theorem I. In Section 3, we prove Theorem I for toral-type Shimura data. In Section 4 we construct Igusa stacks for arbitrary Shimura data of abelian type, using as inputs the results of Section 3 and the results of [Kim25]. In particular, the Cartesian diagram in Theorem I is proved at the end of this section.

In Section 5, we first complete the proof of Theorem I by proving that the Igusa stack is ℓ-cohomologically smooth with constant dualizing sheaf. This section also contains the cohomological results described in Section 1.4.

In Section 6, we study the cohomology of compact Shimura varieties of abelian type and establish its compatibility with Fargues–Scholze parameters for certain classical groups of type A, B, D, and C_2. These compatibilities are then used in the proof of Theorem II, which occupies most of Section 7. We prove Corollary III in Section 7 as well.

Finally, Appendix A is devoted to extending results of [DvHKZ24a, Section 8.2] on canonical Frobenius descent data, using six-functor formalisms.

1.7. **Notation and conventions.**

- A map $f\colon G \to H$ of reductive groups over a field F is an *ad-isomorphism* if $f(Z(G)) \subset Z(H)$ and the induced map $f^{\mathrm{ad}}\colon G^{\mathrm{ad}} \to H^{\mathrm{ad}}$ is an isomorphism.
- Let Perf be the category of perfectoid spaces in characteristic p.
- If T is a topological space, we denote by $\underline{T}$ the sheaf on Perf defined such that $\underline{T}(X)$ is the set of continuous maps from the underlying topological space $|X|$ of X to T.
- For every $X \in \mathrm{Perf}$ we denote by $\phi_X\colon X \to X$ the absolute Frobenius automorphism. This naturally extends to an absolute Frobenius $\phi\colon X \to X$ for an arbitrary v-stack $X \in \mathrm{Stk}(\mathrm{Perf}_{\mathrm{v}})$, and induces $\phi\colon \mathcal{Y}_{[0,\infty)}(R, R^+) \to \mathcal{Y}_{[0,\infty)}(R, R^+)$ as well as $\phi\colon \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$ a lift of the absolute Frobenius.
- For G an algebraic group or a profinite group, by a G-torsor we mean a left G-torsor unless specified otherwise.
- Let $G/\mathbb{Q}_p$ be a connected reductive group. By a G-isocrystal on a perfect $\mathbb{F}_p$-algebra R we mean a G-torsor $\mathscr{P}$ on $\operatorname{Spec} W(R)[1/p]$ together with a morphism $\phi_{\mathscr{P}}\colon \phi^*\mathscr{P} \dashrightarrow \mathscr{P}$. Given an element $b \in G(\breve{\mathbb{Q}}_p)$, we denote by $\mathscr{P}_b$ the G-isocrystal
$$(G_{\breve{\mathbb{Q}}_p}, \phi^* G_{\breve{\mathbb{Q}}_p} \cong G_{\breve{\mathbb{Q}}_p} \xrightarrow{r_{b^{-1}}} G_{\breve{\mathbb{Q}}_p})$$
where $r_{b^{-1}}$ is right multiplication by b^{-1}. We also denote by $\mathscr{P}_b$ the induced map $\operatorname{Spd} \overline{\mathbb{F}}_p \to \mathrm{Bun}_G$, see [GIZ23, Theorem 7.14]. Its automorphism group is given by
$$G_b(\mathbb{Q}_p) = \{g \in G(\breve{\mathbb{Q}}_p) : gb\phi(g)^{-1} = b\},$$
with the right translation action.

- For E/F a Galois extension, we let $\mathrm{Gal}(E/F)$ act from the left on E and act from the right on $\mathrm{Spec}\, E$.
- Given E a local field, we use the geometric normalization for the local Artin map $\mathrm{Art}_E\colon E^\times \hookrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E)$ so that the uniformizer maps to a lift of $\phi^{-[k_E:\mathbb{F}_p]} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/k_E)$. For the global Artin map, we use the normalization that is compatible with the local Artin map, via inclusion in the source and restriction in the target. For example, $\mathrm{Art}_{\mathbb{Q}}\colon \widehat{\mathbb{Z}}^\times \cong \mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{R}_{>0} \to \mathrm{Gal}(\mathbb{Q}(\zeta_\infty)/\mathbb{Q})$ satisfies $\mathrm{Art}_{\mathbb{Q}}(\gamma)(\zeta_n) = \zeta_n^{\gamma \bmod n}$.
- We will denote Shimura data and their reflex fields by sans serif fonts, e.g., (G, X) for a Shimura datum and E for its reflex field. For a prime p and a place v of E above p, we will write E for the completion of E at v, which has ring of integers $\mathcal{O}_E$ and residue field k_E. We will then write G for the base change of G to $\mathbb{Q}_p$.
- Our (conjugacy class of) Hodge cocharacter(s), normalized as in [Kis17, Section 1.3.1], is denoted by μ and is defined over E. We consider this as a $G(\overline{\mathbb{Q}}_p)$-conjugacy class using the place v of E. We use Gr_G to denote the $\mathbf{B}_{\mathrm{dR}}^+$-affine Grassmannian of Scholze–Weinstein, with its stratification by Schubert cells as defined in [SW20, Definition 19.2.2] (this agrees with [PR24] but is the opposite of [CS17]). The Shimura variety over E with infinite level at p has a Hodge–Tate period map with target in $\mathrm{Gr}_{G,\mu^{-1}}$. Its Newton stratification is moreover indexed by the μ^{-1}-admissible set $B(G, \mu^{-1}) \subset B(G)$.

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2. Preliminaries

We work with the category of sheaves (or stacks) on Perf with the v-topology. For a reductive group $G/\mathbb{Q}_p$, we may consider the moduli space Bun_G of G-torsors over the Fargues–Fontaine curve, and given a geometric cocharacter $\mu\colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ the corresponding subset $B(G, \mu^{-1})$ defines an open v-subsheaf $\mathrm{Bun}_{G,\mu^{-1}} \subseteq \mathrm{Bun}_G$. For each scheme or adic space $X/\mathbb{Q}_p$, there is a natural v-sheaf $X^\diamondsuit \to \mathrm{Spa}\,\mathbb{Q}_p$. We refer the reader to [DvHKZ24a, Section 2.1] for the details.

2.1. **Igusa stacks.**

2.1.1. Let (G, X) be a Shimura datum with reflex field E. We choose a place $v \mid p$ of E and write $E = \mathsf{E}_v$ and $G = \mathsf{G}_{\mathbb{Q}_p}$. There is an étale scheme $X_*(\mathsf{G}) /\!\!/ \mathsf{G}$ over $\mathbb{Q}$ that parametrizes cocharacters of G up to conjugacy, where the Hodge cocharacter defines an open and closed embedding $\mu\colon \operatorname{Spec}\mathsf{E} \hookrightarrow X_*(\mathsf{G}) /\!\!/ \mathsf{G}$. Over $\mathbb{Q}_p$, we then get a conjugacy class $\{\mu\colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}\}$ upon fixing an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$.

2.1.2. For each neat open compact subgroup $K \subseteq \mathsf{G}(\mathbb{A}_f)$ there is a Shimura variety $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ over E. We may base change to E and consider the associated v-sheaf $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E^{\diamondsuit}$. We also consider the infinite level Shimura variety

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{\diamondsuit} = \varprojlim_{K\to\{1\}} \mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E^{\diamondsuit},$$

where the limit is taken in the category of v-sheaves. Because the locally profinite group $\mathsf{G}(\mathbb{A}_f)$ acts continuously on the tower, the limit $\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{\diamondsuit}$ carries an action of $\underline{\mathsf{G}(\mathbb{A}_f)}$.

2.1.3. When (G,X) is of abelian type, [IM20] constructs for every neat open compact subgroup $K \subseteq \mathsf{G}(\mathbb{A}_f)$ a quasi-compact open subset

$$\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E^{\circ} \subseteq \mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E^{\mathrm{an}}$$

of the rigid analytification of $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_E$ called the *potentially crystalline locus*. It is uniquely characterized by quasi-compactness together with the property that its classical points are precisely those whose associated p-adic Galois representation is potentially crystalline. The potentially crystalline locus is compatible with pulling back along $K' \hookrightarrow K$, and hence we obtain a quasi-compact open v-subsheaf

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{\circ,\diamondsuit} \subseteq \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{\diamondsuit}.$$

When G is $\mathbb{Q}$-anisotropic modulo center, this inclusion is an equality, see [IM20, Example 5.20].

2.1.4. As discussed in [Han16, Section 5.4], [PR24, Section 2.6], [Kim25, Section 4], there is a $\underline{G(\mathbb{Q}_p)}$-equivariant morphism

$$\pi_{\mathrm{HT}}\colon \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{\diamondsuit} \to \mathrm{Gr}_{G,\mu^{-1}},$$

called the Hodge–Tate period map. We recall also the Beauville–Laszlo map

$$\mathrm{BL}\colon \mathrm{Gr}_{G,\mu^{-1}} \to \mathrm{Bun}_{G,\mu^{-1}},$$

which is surjective in the pro-étale topology, see the proof of [DvHKZ24a, Proposition 6.4.1].

Definition 2.1.5 ([Kim25, Definition 5.5])**.** Let (G,X) be a Shimura datum, choose a place $v \mid p$ of E, and consider $? \in \{\circ, \emptyset\}$. An *Igusa stack* (for ?) is a v-stack $\mathrm{Igs}^?(\mathsf{G},\mathsf{X})$ carrying a $\underline{\mathsf{G}(\mathbb{A}_f^p)}$-action, together with a $\underline{\mathsf{G}(\mathbb{A}_f^p)}$-equivariant map $\bar{\pi}_{\mathrm{HT}}\colon \mathrm{Igs}^?(\mathsf{G},\mathsf{X}) \to \mathrm{Bun}_{G,\mu^{-1}}$, and a 2-Cartesian diagram

$$\begin{tikzcd}
\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \arrow[r, "\pi_{\mathrm{HT}}"] \arrow[d] & \mathrm{Gr}_{G,\mu^{-1}} \arrow[d, "\mathrm{BL}"] \\
\mathrm{Igs}^?(\mathsf{G},\mathsf{X}) \arrow[r, "\bar{\pi}_{\mathrm{HT}}"] & \mathrm{Bun}_{G,\mu^{-1}}
\end{tikzcd}$$

such that the induced isomorphism

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \cong \mathrm{Igs}^?(\mathsf{G},\mathsf{X}) \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}}$$

is equivariant for the natural $\mathsf{G}(\mathbb{A}_f)$-actions on both sides, and $\phi \times \mathrm{id}$ acts trivially on the right hand side.

Oftentimes, it is more convenient to encode the information of the Igusa stack as a single map between v-sheaves.

Definition 2.1.6 ([Kim25, Definition 5.11])**.** In the setting of Definition 2.1.5, a *global uniformization* is a map of v-sheaves

$$\Theta\colon \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \to \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit}$$

such that

(1) the map Θ is $\mathsf{G}(\mathbb{A}_f) \times G(\mathbb{Q}_p)$-equivariant, where the action on $\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit}$ is given via $\mathsf{G}(\mathbb{A}_f) \times G(\mathbb{Q}_p) = \mathsf{G}(\mathbb{A}_f^p) \times G(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \xrightarrow{\mathrm{pr}_{13}} \mathsf{G}(\mathbb{A}_f^p) \times G(\mathbb{Q}_p) = \mathsf{G}(\mathbb{A}_f)$,
(2) the map Θ intertwines the $\phi \times \mathrm{id}$-action on the source and the trivial action on the target,
(3) the diagrams

$$\begin{tikzcd} \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \arrow[rr, equal] \arrow[d, "{(\mathrm{id},\pi_{\mathrm{HT}})}"] & & \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \arrow[d, "\pi_{\mathrm{HT}}"] \\ \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \arrow[urr, "\Theta"] \arrow[rr, "\mathrm{pr}_2"] & & \mathrm{Gr}_{G,\mu^{-1}} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \arrow[r, "\Theta\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_{13}"] & \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \arrow[d, "\Theta"] \\ \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \arrow[r, "\Theta"] & \mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \end{tikzcd}$$

commute.

We may now summarize the results of [Kim25] as follows.

Theorem 2.1.7. *Let* (G,X) *be a Shimura datum with reflex field* E.

(*i*) *The existence of an Igusa stack* $\mathrm{Igs}^?(\mathsf{G},\mathsf{X})$ *is equivalent to the existence of a global uniformization* Θ.
(*ii*) *If there exists an Igusa stack* $\mathrm{Igs}^?(\mathsf{G},\mathsf{X})$*, then it is unique up to unique isomorphism. Equivalently, if there exists a global uniformization* Θ*, then it is unique.*
(*iii*) *Let* $(\mathsf{G},\mathsf{X}) \to (\mathsf{G}',\mathsf{X}')$ *be a morphism of Shimura data inducing a morphism* $\mathbf{Sh}(\mathsf{G},\mathsf{X})_E^{?,\diamondsuit} \to \mathbf{Sh}(\mathsf{G}',\mathsf{X}')_{E'}^{?,\diamondsuit}$*. If there exist Igusa stacks* $\mathrm{Igs}^?(\mathsf{G},\mathsf{X})$ *and* $\mathrm{Igs}^?(\mathsf{G}',\mathsf{X}')$*, then there exists an induced map* $\mathrm{Igs}^?(\mathsf{G},\mathsf{X}) \to \mathrm{Igs}^?(\mathsf{G}',\mathsf{X}')$ *which*

is unique up to unique isomorphism. Equivalently, the global uniformization maps fit in a commutative diagram

$$\begin{array}{ccc}
\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{E} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} & \xrightarrow{\Theta} & \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{E} \\
\downarrow & & \downarrow \\
\mathbf{Sh}(\mathsf{G}',\mathsf{X}')^{?,\diamondsuit}_{E'} \times_{\mathrm{Bun}_{G',\mu'^{-1}}} \mathrm{Gr}_{G',\mu'^{-1}} & \xrightarrow{\Theta'} & \mathbf{Sh}(\mathsf{G}',\mathsf{X}')^{?,\diamondsuit}_{E'}.
\end{array}$$

2.2. **Locally profinite groups.**

2.2.1. We recall that a short exact sequence of locally profinite groups is an (algebraic) short exact sequence

$$1 \to H \to G \to K \to 1$$

where $H \subseteq G$ is closed with respect to the subspace topology, and $G \to K$ has the quotient topology. Note that every closed subgroup of a locally profinite group G is locally profinite, and given such a closed subgroup $H \subseteq G$, the quotient G/H with its quotient topology is a locally profinite set.

Proposition 2.2.2. *For a short exact sequence* $1 \to H \to G \to K \to 1$ *of locally profinite groups, the induced sequence*

$$1 \to \underline{H} \to \underline{G} \to \underline{K} \to 1$$

of presheaves on Perf(*hence of pro-étale sheaves or v-sheaves*) *is short exact.*

Proof. It is clear that the sequence is left exact, and it remains to verify surjectivity of $\underline{G} \to \underline{K}$. This follows from the fact that $G \to K$ admits a continuous section, see [RZ10, Proposition 2.2.2]. ☐

2.2.3. Let G be a locally profinite group. There is a corresponding v-sheaf of automorphisms $\mathrm{Aut}(\underline{G})$, and its action on the 1-dimensional $\mathbb{Q}$-vector space $\mathrm{Haar}(G,\mathbb{Q})$ of right Haar measures defines a character

$$\delta_G\colon \mathrm{Aut}(\underline{G}) \to \underline{\mathbb{Q}^\times_{>0}}, \quad \delta_G(\sigma)\mu(\sigma(U)) = \mu(U),$$

see Lemma 2.2.4. If G is moreover unimodular, then the inner automorphisms act trivially on $\mathrm{Haar}(G,\mathbb{Q})$, and hence we obtain

$$\delta_G\colon \mathrm{Out}(\underline{G}) \coloneqq \mathrm{Aut}(\underline{G})/\underline{G} \to \underline{\mathbb{Q}^\times_{>0}}.$$

Lemma 2.2.4. *Let* X *be a topological space and* G *be a locally profinite group. Let* $f\colon X \times G \to X \times G$ *be a homeomorphism over* X *with the property that* $f_x\colon G \to G$ *is a group automorphism for every* $x \in X$*. Then the function*

$$\delta_G(f)\colon X \to \mathbb{Q}^\times_{>0}; \quad x \mapsto \delta_G(f_x)$$

is locally constant.

Proof. Fix $x \in X$, fix a compact open $K \subseteq G$, and write $L = f_x(K)$. It suffices to find a neighborhood $U \ni x$ such that $f(U \times K) = U \times L$. Because $f(X \times K)$ is open and contains $\{x\} \times L$, it contains $U_1 \times L$ for some open neighborhood $U_1 \ni x$ since L is compact. Similarly, $f^{-1}(X \times L)$ contains $U_2 \times K$ for some neighborhood $U_2 \ni x$. We now observe that $U = U_1 \cap U_2$ does the trick. $\square$

Example 2.2.5. When G is profinite, every automorphism of G preserves every Haar measure since the Haar measure μ can be normalized to satisfy $\mu(G) = 1$. It follows that $\delta_G = 1$. There are unimodular groups G with non-trivial δ_G. For example, $G = \mathbb{Q}_p$ has

$$\delta_G \colon \operatorname{Out}(\underline{G}) \cong \underline{\mathbb{Q}_p^\times} \to p^{\mathbb{Z}}, \quad \delta_G(x) = p^{v_p(x)}.$$

2.3. **An integral conjecture.** Fix (G, X) and $v \mid p$ as before. Let $\mathcal{G}$ be a parahoric model of G. Recall the cuspidal quotient $\mathsf{G} \to \mathsf{G}^c$ as in [KSZ21, 1.5.8], which induces a parahoric model $\mathcal{G}^c$ of $G^c = (\mathsf{G}^c)_{\mathbb{Q}_p}$ (see [DY25, Section 4.1.1]). There is a (conjectural) canonical integral model $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})$ of $\mathbf{Sh}_{K_p}(\mathsf{G}, \mathsf{X})$ over $\mathcal{O}_E$, characterized by a morphism

$$\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamondsuit/} \to \operatorname{Sht}_{\mathcal{G}^c, \mu^c},$$

see [Dan25, Conjecture 4.5] and [PR24, Conjecture 4.2.2]. The existence is known when $p \geq 3$ and (G, X) is of abelian type, see [DY25], or when (G, X) is of Hodge type, see [PR24] and [DvHKZ24b].

Conjecture 2.3.1. *Assume that* (G, X) *admits an Igusa stack* $\operatorname{Igs}^\circ(\mathsf{G}, \mathsf{X})$ *at* v. *Given a parahoric model* $\mathcal{G}$ *of* G *with* $K_p = \mathcal{G}(\mathbb{Z}_p)$, *we consider the fiber product*

$$\begin{tikzcd} S \arrow[r, "\pi_{\mathrm{crys}}"] \arrow[d] & \operatorname{Sht}_{\mathcal{G},\mu} \arrow[d, "\mathrm{BL}^\circ"] \\ \operatorname{Igs}^\circ(\mathsf{G}, \mathsf{X}) \arrow[r, "\overline{\pi}_{\mathrm{HT}}"] & \operatorname{Bun}_{G,\mu^{-1}}. \end{tikzcd}$$

Then the composition $S \to \operatorname{Sht}_{\mathcal{G},\mu} \to \operatorname{Sht}_{\mathcal{G}^c,\mu^c}$ *is* $\underline{\mathsf{G}(\mathbb{A}_f^p)}$*-equivariantly isomorphic to* $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamond} \to \operatorname{Sht}_{\mathcal{G}^c,\mu^c}$.

If (G, X) is of Hodge type, then $\mathsf{G} = \mathsf{G}^c$ and this is [DvHKZ24a, Theorem VII].

Remark 2.3.2. One could also ask for an analogue of Conjecture 2.3.1 for $\operatorname{Igs}(\mathsf{G}, \mathsf{X})$ instead of $\operatorname{Igs}^\circ(\mathsf{G}, \mathsf{X})$. In this case, we expect the fiber product to be $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamondsuit}$. The existence of the map $\mathscr{S}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamondsuit} \to \operatorname{Sht}_{\mathcal{G}^c,\mu^c}$ is being investigated in ongoing work [MW26] of Mao–Wu.

Remark 2.3.3. Recall that the generic fiber $\operatorname{Sht}_{\mathcal{G},\mu} \times_{\operatorname{Spd} \mathcal{O}_E} \operatorname{Spd} E$ is isomorphic to the quotient stack (see [Zha23, Proposition 11.16])

$$\left[\operatorname{Gr}_{G,\mu^{-1}} / \underline{K_p}\right].$$

Since we have a $G(\mathbb{Q}_p)$-equivariant Hodge–Tate period map $\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{\diamondsuit} \to \operatorname{Gr}_{G,\mu^{-1}}$, the existence of the map $\mathbf{Sh}_{K_p}(\mathsf{G}, \mathsf{X})^{\diamondsuit} \to \operatorname{Sht}_{\mathcal{G},\mu}$ on the generic fiber comes down to $\mathbf{Sh}(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_{K_p}(\mathsf{G},\mathsf{X})$ being a K_p-torsor. To see this, it suffices to verify that the closure $Z_{\mathsf{G}}(\mathbb{Q})^- \subseteq \mathsf{G}(\mathbb{A}_f)$ intersects $G(\mathbb{Q}_p)$ trivially, see [KSZ21, Section 1.5.8]. This assertion is verified in Lemma 2.3.4 below. Note that at finite level, the map $\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ is not necessarily a K_p-torsor, and hence we cannot generally expect a $\mathcal{G}$-shtuka over $\mathscr{S}_K(\mathsf{G},\mathsf{X})^\diamond$.

Lemma 2.3.4. *Let $\mathsf{H}/\mathbb{Q}$ be a multiplicative (not necessarily connected) group. Then the closure $\mathsf{H}(\mathbb{Q})^-$ of $\mathsf{H}(\mathbb{Q}) \subseteq \mathsf{H}(\mathbb{A}_f)$ intersects $\prod_{p\in S} \mathsf{H}(\mathbb{Q}_p)$ trivially for every finite set of primes S.*

Proof. This easily follows from the congruence subgroup property for tori: Upon choosing a set of generators of $X^*(\mathsf{H})$, we obtain an embedding of $\mathsf{H} \hookrightarrow \prod_{i=1}^N \mathrm{Res}_{F_i/\mathbb{Q}}\, \mathbb{G}_m$ for finite extensions $F_i/\mathbb{Q}$. Then the problem reduces to the case when $\mathsf{H} = \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_m$. At this point, we are considering the closure of $F^\times \hookrightarrow \mathbb{A}_{F,f}^\times$. Once we intersect with the open subset $U = \prod_{p\notin S} \mathcal{O}_{F,p}^\times \times \prod_{p\in S} F_p^\times \subseteq \mathbb{A}_{F,f}^\times$, we get

$$(F^\times)^- \cap U = (\text{closure of } \mathcal{O}_{F,S}^\times \hookrightarrow U),$$

where the group of S-units $\mathcal{O}_{F,S}^\times$ is a finitely generated abelian group. Applying [Che51, Theorem 1] to $\mathcal{O}_{F,S}^\times$, we see that for every integer $m \geq 1$ there exists an open subgroup $U' = U^{S\prime} \times \prod_{p\in S} F_p^\times \subseteq U$ for which $U' \cap \mathcal{O}_{F,S}^\times \subseteq (\mathcal{O}_{F,S}^\times)^m$. Taking the intersection over all m, we obtain the desired result. ☐

3. Igusa stacks for tori

In this section we prove Theorem I for Shimura data (G,X) with G a torus. To this end, we give a definition of the Igusa stack in Section 3.1. We prove that our definition satisfies Theorem I in Section 3.3, which relies on an explicit description of the Beauville–Laszlo map for tori in Section 3.2.

3.1. Construction of the Igusa stack.

3.1.1. Let $G/\mathbb{Q}_p$ be a connected reductive group and let $b \in G(\breve{\mathbb{Q}}_p)$ be an element. We consider the group

$$A_{G,b} = \{(x,\sigma) \in G(\breve{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p) : x^{-1} b \phi(x) = \sigma(b)\},$$

which is the centralizer of the element (b,ϕ). This is a closed subgroup of $G(\breve{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p)$, and fits in a short exact sequence

$$1 \to G_b(\mathbb{Q}_p) \to A_{G,b} \to \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p) \cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1$$

of locally profinite groups, where $G_b(\mathbb{Q}_p) = \{(x,\sigma) : x^{-1} b \phi(x) = b\}$. We also consider the subgroup $A_{G,b}^0 \subset A_{G,b}$ generated by $G_b(\mathbb{Q}_p)$ and (b,ϕ), which can also be described as the preimage of $\phi^{\mathbb{Z}} \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The element (b,ϕ) induces a splitting

$$A_{G,b}^0 \cong G_b(\mathbb{Q}_p) \times \mathbb{Z}.$$

Lemma 3.1.2. *There exists an open subgroup $U \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ over which there exists a section $U \to A_{G,b}$. Moreover, the locally profinite group $A_{G,b}$ is the closure of the subgroup $A^0_{G,b}$ in $G(\breve{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. In particular, if $G = T$ is a torus, then $A_{T,b}$ is abelian.*

Proof. For the first claim, we use [Kot85, Section 4.3] to find an element $g_0 \in G(\breve{\mathbb{Q}}_p)$ for which $g_0^{-1} b\phi(g_0) \in G(\mathbb{Q}_{p^n})$ for some positive integer n. Then we see that

$$U = \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_{p^n}) \ni \sigma \mapsto (g_0\sigma(g_0)^{-1}, \sigma) \in A_{G,b}$$

is a section, where we compute that $\sigma(g_0^{-1}b\phi(g_0)) = g_0^{-1}b\phi(g_0)$ implies

$$(g_0\sigma(g_0)^{-1})^{-1} b\phi(g_0\sigma(g_0)^{-1}) = \sigma(b).$$

Such a section induces a homeomorphism between $A_{G,b}|_U$ and $U \times G_b(\mathbb{Q}_p)$, under which the subset $A^0_{G,b}|_U$ corresponds to $(U \cap \phi^{\mathbb{Z}}) \times G_b(\mathbb{Q}_p)$. This is dense, and hence the closure of $A^0_{G,b} \subseteq G(\breve{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is $A_{G,b}$.

Finally, if $G = T$ is a torus, then $G_b(\mathbb{Q}_p) = T(\mathbb{Q}_p)$. As $T(\mathbb{Q}_p) \subseteq T(\breve{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\breve{\mathbb{Q}}_p/\mathbb{Q}_p)$ is central, the subgroup $A^0_{T,b}$ generated by $T(\mathbb{Q}_p)$ and (b, ϕ) is abelian. It follows that its closure $A_{T,b}$ is abelian as well. □

3.1.3. Given a ϕ-conjugate $gb\phi(g)^{-1}$ of b, where $g \in G(\breve{\mathbb{Q}}_p)$ is an arbitrary element, there is an isomorphism of locally profinite groups

$$c_g \colon A_{G,b} \xrightarrow{\cong} A_{G,gb\phi(g)^{-1}}; \quad (x, \sigma) \mapsto (gx\sigma(g)^{-1}, \sigma).$$

Under this isomorphism, we observe that $(b, \phi) \in A_{G,b}$ is mapped to $(gb\phi(g)^{-1}, \phi)$. It is also clear that for a homomorphism $f \colon G \to H$ of connected reductive groups and $b \in G(\breve{\mathbb{Q}}_p)$, there is a natural continuous group homomorphism

$$f \colon A_{G,b} \to A_{H,f(b)}$$

that respects the projection to $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

3.1.4. We may also interpret the locally profinite groups $A_{G,b}$ as follows. By [Ans23, Theorem 5.3], every element $b \in G(\breve{\mathbb{Q}}_p)$ defines a map $b \colon \mathrm{Spd}\,\overline{\mathbb{F}}_p \to \mathrm{Bun}_G$. We may consider the automorphism group of this map, regarded as a functor on perfectoid $\mathbb{F}_p$-algebras. More precisely, let (R, R^+) be a perfectoid $\mathbb{F}_p$-algebra that admits an $\overline{\mathbb{F}}_p$-algebra structure $f \colon \overline{\mathbb{F}}_p \to R^+$. Any other $\overline{\mathbb{F}}_p$-structure may be obtained by conjugating f by an element $\sigma \in \underline{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}(R, R^+)$, i.e., as $f \circ \sigma \colon \overline{\mathbb{F}}_p \to (R, R^+)$. Both of these $\overline{\mathbb{F}}_p$-algebra structures give rise to G-torsors $f^*\mathscr{P}_b$ and $(f \circ \sigma)^*\mathscr{P}_b$ on $\mathcal{X}_{\mathrm{FF}}(R, R^+)$, see Section 1.7. Given a lift of σ to a continuous map $x \colon |\mathrm{Spa}(R, R^+)| \to A_{G,b}$ we obtain an isomorphism

$$f^*\mathscr{P}_b \xrightarrow{\cong} (f \circ \sigma)^*\mathscr{P}_b$$

given by ϕ-descending right multiplication $r_{f(x)}$ by $f(x) \in G(\mathcal{Y}_{(0,\infty)}(R, R^+))$. (The map f defines a structure map $\mathcal{Y}_{(0,\infty)}(R, R^+) \to \mathrm{Spd}\,\breve{\mathbb{Q}}_p$, and we denote by $f(x) \in$

$G(\mathcal{Y}_{(0,\infty)}(R,R^+))$ the pullback of $x \in G(\breve{\mathbb{Q}}_p)$ along this structure map.) This is because the diagram

$$\begin{array}{ccc} \phi^* G_{\mathcal{Y}_{(0,\infty)}(R,R^+)} & \xrightarrow{r_{f(\phi(x))}} & \phi^* G_{\mathcal{Y}_{(0,\infty)}(R,R^+)} \\ {\scriptstyle r_{f(b)^{-1}}}\downarrow & & \downarrow{\scriptstyle r_{f(\sigma(b))^{-1}}} \\ G_{\mathcal{Y}_{(0,\infty)}(R,R^+)} & \xrightarrow{r_{f(x)}} & G_{\mathcal{Y}_{(0,\infty)}(R,R^+)} \end{array}$$

commutes, as we have $x^{-1}b\phi(x) = \sigma(b)$ from Section 3.1.1. This construction defines a morphism of v-stacks

$$[\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b}}] \to \operatorname{Bun}_G^{[b]} \subseteq \operatorname{Bun}_G,$$

where the right $A_{G,b}$-action on $\operatorname{Spd} \overline{\mathbb{F}}_p$ is through $A_{G,b} \twoheadrightarrow \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

Proposition 3.1.5. *Assume that $[b] \in B(G)$ is basic. Then the map*

$$[\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b}}] \xrightarrow{\cong} \operatorname{Bun}_G^{[b]}$$

constructed in Section 3.1.4 is an isomorphism of v-stacks.

Proof. As $\operatorname{Spd} \overline{\mathbb{F}}_p \to \operatorname{Spd} \mathbb{F}_p$ is a v-cover, we may check that the map is an isomorphism after base changing to $\overline{\mathbb{F}}_p$. Upon identifying $\operatorname{Spd} \overline{\mathbb{F}}_p \times \operatorname{Spd} \overline{\mathbb{F}}_p \cong \operatorname{Spd} \overline{\mathbb{F}}_p \times \underline{\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}$, this boils down to checking that

$$[\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{G_b(\mathbb{Q}_p)}] \to \operatorname{Bun}_{G,\overline{\mathbb{F}}_p}^{[b]}$$

is an isomorphism. This is done in [FS24, Theorem III.0.2]. $\square$

Remark 3.1.6. Even when b is not basic, a similar description of $\operatorname{Bun}_G^{[b]}$ as a quotient of $\operatorname{Spd} \overline{\mathbb{F}}_p$ can be obtained by using an extension of $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ by $\tilde{G}_b$.

3.1.7. Given elements $b, g, b' = gb\phi(g)^{-1} \in G(\breve{\mathbb{Q}}_p)$, the element g induces an isomorphism between $b, b' \colon \operatorname{Spd} \overline{\mathbb{F}}_p \to \operatorname{Bun}_G$, and there is an isomorphism $c_g \colon A_{G,b} \cong A_{G,b'}$ from Section 3.1.3. Assume from now on that $[b] = [b'] \in B(G)$ is basic. It follows from the construction of Section 3.1.4 that the isomorphism

$$[\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b}}] \cong \operatorname{Bun}_G^{[b]} = \operatorname{Bun}_G^{[b']} \cong [\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b'}}]$$

agrees with $[\mathrm{id}/c_g]$. We also recall from [DvHKZ24a, Proposition 2.1.10] that there exists a canonical isomorphism between $\phi, \mathrm{id} \colon \operatorname{Bun}_G \to \operatorname{Bun}_G$, and hence between $\phi, \mathrm{id} \colon \operatorname{Bun}_G^{[b]} \to \operatorname{Bun}_G^{[b]}$. Using the presentation $\operatorname{Bun}_G^{[b]} \cong [\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b}}]$, we see that the absolute Frobenius ϕ can be presented as

$$\phi = [\phi_{\overline{\mathbb{F}}_p}/\mathrm{id}] \colon [\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b}}] \xrightarrow{\cong} [\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{G,b}}].$$

Unraveling the construction, we verify that the isomorphism from id to ϕ is given as right multiplication by the element (b, ϕ), where we note that $(b, \phi) \in A_{G,b}$ is central.

3.1.8. Let $\mathsf{T}/\mathbb{Q}$ be a torus, and write $T = \mathsf{T}_{\mathbb{Q}_p}$. For each $b \in T(\breve{\mathbb{Q}}_p)$, we have an extension of locally profinite abelian groups

$$1 \to T(\mathbb{Q}_p) \to A_{T,b} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1.$$

We also have a profinite group $\mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$ and a continuous group homomorphism

$$\iota\colon T(\mathbb{Q}_p) \hookrightarrow \mathsf{T}(\mathbb{A}_f) \twoheadrightarrow \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f).$$

Lemma 3.1.9. *There exists a unique continuous group homomorphism*

$$\iota_{T,b}\colon A_{T,b} \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$$

with the property that $\iota_{T,b}(b,\phi) = 1$ and the restriction of $\iota_{T,b}$ to $T(\mathbb{Q}_p) \subset A_{T,b}$ recovers the inclusion ι.

Proof. Because $A^0_{T,b}$ is the direct sum of $(b,\phi)^{\mathbb{Z}}$ and $T(\mathbb{Q}_p)$, we have a group homomorphism

$$\iota^0_{T,b}\colon A^0_{T,b} \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f); \quad (b,\phi) \mapsto 1, \quad (t,1) \mapsto \iota(t)$$

that we wish to extend to $A_{T,b}$. Uniqueness follows from the density of $A^0_{T,b} \subset A_{T,b}$, see Lemma 3.1.2. For existence, we use Lemma 3.1.2 to find an open subgroup $U = n\,\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ over which there exists an isomorphism $A_{T,b}|_U \cong U \oplus T(\mathbb{Q}_p)$ of topological abelian groups. This restricts to $A^0_{T,b}|_U \cong \phi^{n\mathbb{Z}} \oplus T(\mathbb{Q}_p)$, and because $\mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$ is profinite, any group homomorphism $\phi^{n\mathbb{Z}} \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$ uniquely extends to a continuous group homomorphism $U \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$. This gives a continuous extension

$$A_{T,b}|_U \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$$

of $\iota^0_{T,b}|_U$. We now combine the two group homomorphisms, one on $A_{T,b}|_U$ and one on $A^0_{T,b}$, to a group homomorphism on $A_{T,b}$ using

$$A_{T,b}|_U + A^0_{T,b} = A_{T,b}, \quad A_{T,b}|_U \cap A^0_{T,b} = A^0_{T,b}|_U.$$

The resulting group homomorphism $\iota_{T,b}\colon A_{T,b} \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$ is indeed continuous because its restriction to the open subgroup $A_{T,b}|_U$ is continuous. $\square$

3.1.10. Recall that given $b \in T(\breve{\mathbb{Q}}_p)$ and $b' = gb\phi(g)^{-1}$ there is a canonical isomorphism $c_g\colon A_{T,b} \cong A_{T,b'}$. Because T is commutative, the restriction of c_g to $T(\mathbb{Q}_p)$ is the identity map. It follows from uniqueness above that we have a commutative diagram

$$\begin{tikzcd}
A_{T,b} \arrow[r, "\iota_{T,b}"] \arrow[d, "c_g"', "\cong"] & \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f) \arrow[d, equal] \\
A_{T,b'} \arrow[r, "\iota_{T,b'}"] & \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f).
\end{tikzcd}$$

Definition 3.1.11. For each $b \in T(\breve{\mathbb{Q}}_p)$ corresponding to a map $\operatorname{Spd} \overline{\mathbb{F}}_p \to \mathrm{Bun}_T$, hence inducing an identification $\mathrm{Bun}_T^{[b]} \cong [\operatorname{Spd} \overline{\mathbb{F}}_p/\underline{A_{T,b}}]$, we define the v-stack

$$\mathrm{Igs}(\mathsf{T}, b) = [(\underline{\mathsf{T}(\mathbb{Q})^{-}\backslash \mathsf{T}(\mathbb{A}_f)} \times \operatorname{Spd} \overline{\mathbb{F}}_p)/\underline{A_{T,b}}] \to [\operatorname{Spd} \overline{\mathbb{F}}_p/\underline{A_{T,b}}] \cong \mathrm{Bun}_T^{[b]},$$

where $(x, \sigma) \in A_{T,b}$ acts on $\mathsf{T}(\mathbb{Q})^{-}\backslash \mathsf{T}(\mathbb{A}_f)$ via translation by $\iota_{T,b}(x, \sigma)$ and on $\operatorname{Spd} \overline{\mathbb{F}}_p$ via σ.

3.1.12. Write $b' = gb\phi(g)^{-1}$ for some $g \in T(\breve{\mathbb{Q}}_p)$. We see from Section 3.1.10 that the action of $\underline{A_{T,b}}$ on $\underline{\mathsf{T}(\mathbb{Q})^{-}\backslash \mathsf{T}(\mathbb{A}_f)} \times \operatorname{Spd} \overline{\mathbb{F}}_p$ is compatible with the action of $A_{T,b'}$ under the identification $c_g \colon A_{T,b} \cong A_{T,b'}$. It follows that there is a natural commutation relation

$$\begin{tikzcd}
\mathrm{Igs}(\mathsf{T}, b) \arrow[r] \arrow[d, "{[\mathrm{id}\times\mathrm{id}/c_g]}"', "\cong"] & {[\operatorname{Spd} \overline{\mathbb{F}}_p/\underline{A_{T,b}}]} \arrow[dr, "\cong"] \arrow[d, "{[\mathrm{id}/c_g]}"', "\cong"] & \\
\mathrm{Igs}(\mathsf{T}, b') \arrow[r] & {[\operatorname{Spd} \overline{\mathbb{F}}_p/\underline{A_{T,b'}}]} \arrow[r, "\cong"] & \mathrm{Bun}_T^{[b]}
\end{tikzcd}$$

that is moreover associative in the sense that it is compatible with a further ϕ-conjugation $b'' = g'b'\phi(g')^{-1}$. Using these identifications, we may regard $\mathrm{Igs}(\mathsf{T}, b)$ as depending only on the ϕ-conjugacy class $[b] \in B(T)$.

Definition 3.1.13. We define the *Igusa stack* for $\mathsf{T}/\mathbb{Q}$ as the stack

$$\mathrm{Igs}(\mathsf{T}) = \coprod_{[b]\in B(T)} \mathrm{Igs}(\mathsf{T}, b) \to \coprod_{[b]\in B(T)} \mathrm{Bun}_T^{[b]} = \mathrm{Bun}_T.$$

Remark 3.1.14. We remark that this does not depend on the choice of the cocharacter μ nor the choice of a place $v \mid p$ of the reflex field. See also Remark 3.3.8.

3.1.15. For every element $t \in \mathsf{T}(\mathbb{A}_f^p)$, there is an induced map

$$t \colon [(\underline{\mathsf{T}(\mathbb{Q})^{-}\backslash \mathsf{T}(\mathbb{A}_f)} \times \operatorname{Spd} \overline{\mathbb{F}}_p)/\underline{A_{T,b}}] \to [(\underline{\mathsf{T}(\mathbb{Q})^{-}\backslash \mathsf{T}(\mathbb{A}_f)} \times \operatorname{Spd} \overline{\mathbb{F}}_p)/\underline{A_{T,b}}]$$

given by translation by t^{-1} on the abelian group $\mathsf{T}(\mathbb{Q})^{-}\backslash \mathsf{T}(\mathbb{A}_f)$. This defines a left action of $\underline{\mathsf{T}(\mathbb{A}_f^p)}$ on $\mathrm{Igs}(\mathsf{T}, b)$, where this action does not change when we replace b with a ϕ-conjugate. Moreover, the natural map

$$\mathrm{Igs}(\mathsf{T}, b) \to \mathrm{Bun}_T^{[b]}$$

is naturally $\underline{\mathsf{T}(\mathbb{A}_f^p)}$-equivariant, where we give the trivial action on $\mathrm{Bun}_T^{[b]}$.

3.1.16. As in Section 3.1.7 we may define an isomorphism

$$\mathrm{id}_{\mathrm{Igs}(\mathsf{T},b)} \xrightarrow{\cong} \phi_{\mathrm{Igs}(\mathsf{T},b)}$$

given by right multiplication by the element $(b, \phi) \in A_{T,b}$. This is compatible with the isomorphism $\mathrm{id}_{\mathrm{Bun}_T} \cong \phi_{\mathrm{Bun}_T}$. More precisely, the diagram

$$\begin{tikzcd} \mathrm{Igs}(\mathsf{T}) \arrow[r] \arrow[d, bend right, "\phi"'] \arrow[d, bend left, "\mathrm{id}"] & \mathrm{Bun}_T \arrow[d, bend right, "\phi"'] \arrow[d, bend left, "\mathrm{id}"] \\ \mathrm{Igs}(\mathsf{T}) \arrow[r] & \mathrm{Bun}_T \end{tikzcd}$$

is 2-commutative, i.e., the two isomorphisms between $\mathrm{Igs}(\mathsf{T}) \xrightarrow{\phi} \mathrm{Igs}(\mathsf{T}) \to \mathrm{Bun}_T$ and $\mathrm{Igs}(\mathsf{T}) \to \mathrm{Bun}_T \xrightarrow{\mathrm{id}} \mathrm{Bun}_T$ agree.

3.2. **The Beauville–Laszlo map.**

3.2.1. Let T be a torus over $\mathbb{Q}_p$, and let $\mu\colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to T_{\overline{\mathbb{Q}}_p}$ be a geometric cocharacter whose field of definition is $E \subseteq \overline{\mathbb{Q}}_p$. Writing $[b] \in B(T, \mu^{-1})$ for the unique element, we have a map

$$\mathrm{BL}_{\mu^{-1}}\colon \mathrm{Gr}_{T,\mu^{-1}} \cong \operatorname{Spd} E \to \mathrm{Bun}_T^{[b]} \cong [\operatorname{Spd} \overline{\mathbb{F}}_p / \underline{A_{T,b}}]$$

that sends a perfectoid E-algebra $(R^\sharp, R^{\sharp+})$ to the T-torsor on the Fargues–Fontaine curve $\mathcal{X}_{\mathrm{FF}}(R, R^+)$ obtained by modifying the trivial T-torsor along the specified untilt by μ^{-1}. Our goal is to understand the map $\mathrm{BL}_{\mu^{-1}}$ in terms of group theory.

3.2.2. Let us write $S = \operatorname{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m$. There is a decomposition $E \otimes_{\mathbb{Q}_p} E \cong E \times \prod_i L_i$ where the projection $E \otimes_{\mathbb{Q}_p} E \twoheadrightarrow E$ is given by multiplication. It follows that there is a canonical group homomorphism $E^\times \to (E \otimes_{\mathbb{Q}_p} E)^\times = E^\times \times \prod_i L_i^\times$ given by $x \mapsto (x, 1, \ldots, 1)$, which can be promoted to a morphism

$$\mu_S\colon \mathbb{G}_{m,E} \to (\operatorname{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m)_E = S_E$$

of E-tori. As μ has field of definition E, there is an induced map

$$r_\mu\colon S = \operatorname{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m \to T$$

of $\mathbb{Q}_p$-tori, where $(r_\mu)_E \circ \mu_S$ recovers μ.

3.2.3. Let $E_0 \subseteq E$ be the maximal subfield unramified over $\mathbb{Q}_p$, and let $\breve{E} = E\breve{\mathbb{Q}}_p$. Choose a uniformizer π_E of E. The homomorphism r_μ induces a map

$$B(S, \mu_S^{-1}) \xrightarrow{r_\mu} B(T, \mu^{-1}),$$

where both sides are singletons. As in [Kot85, Section 2.5], we may and will choose

$$b_S = (\pi_E^{-1}, 1, \ldots, 1) \in \prod_{E_0 \hookrightarrow \breve{\mathbb{Q}}_p} \breve{E}^\times = S(\breve{\mathbb{Q}}_p)$$

so that $[b_S] \in B(S, \mu_S^{-1})$, and use its image $b = r_\mu(b_S) \in T(\breve{\mathbb{Q}}_p)$ as a representative of $[b] \in B(T, \mu^{-1})$. We note that b_S may also be understood as the norm $\mathrm{Nm}_{E/E_0}(\mu_S(\pi_E^{-1})) \in S(E_0) \subseteq S(\breve{\mathbb{Q}}_p)$, and hence b may be described as $b = \mathrm{Nm}_{E/E_0}(\mu(\pi_E^{-1})) \in T(E_0) \subseteq T(\breve{\mathbb{Q}}_p)$. We also compute

$$b_S\phi(b_S)\cdots\phi^{f-1}(b_S) = \pi_E^{-1} \in S(\mathbb{Q}_p) \subseteq S(\breve{\mathbb{Q}}_p).$$

where $f = [E_0 : \mathbb{Q}_p]$ is the inertial degree and $\phi\colon S(\breve{\mathbb{Q}}_p) \to S(\breve{\mathbb{Q}}_p)$ is the Frobenius on $S(\breve{\mathbb{Q}}_p)$.

3.2.4. Since $S = \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m$, a left S-torsor over $\mathcal{X}_{\mathrm{FF}}(R,R^+)$ is equivalently a line bundle on the adic space

$$\mathcal{X}_{\mathrm{FF}}(R,R^+)_E = \mathcal{X}_{\mathrm{FF}}(R,R^+) \otimes_{\mathrm{Spa}\,\mathbb{Q}_p} \mathrm{Spa}\,E.$$

More precisely, given a vector bundle $\mathscr{V}$ of rank n on $\mathcal{X}_{\mathrm{FF}}(R,R^+)_E$, its pushforward $\pi_*\mathscr{V}$ along $\pi\colon \mathcal{X}_{\mathrm{FF}}(R,R^+)_E \to \mathcal{X}_{\mathrm{FF}}(R,R^+)$ is a vector bundle with an E-action, and the sheaf of E-linear isomorphisms

$$\mathscr{P}(\mathscr{V}) = \underline{\mathrm{Isom}}_E(\pi_*\mathscr{V}, \mathscr{O}^{\oplus n}_{\mathcal{X}_{\mathrm{FF}}(R,R^+)} \otimes_{\mathbb{Q}_p} E)$$

is a left $\mathrm{Res}_{E/\mathbb{Q}_p}\mathrm{GL}_n$-torsor.

3.2.5. We now construct two left S-torsors on the Fargues–Fontaine curve. Let k_E be the residue field of E. For every perfectoid k_E-algebra (R,R^+), we have a morphism $\mathcal{Y}_{(0,\infty)}(R,R^+)_E \to \mathrm{Spa}(E_0 \otimes_{\mathbb{Q}_p} E)$. Hence we may define the line bundle

$$\mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(R,R^+)_E}(b_S) = \mathscr{O}_{\mathcal{Y}_{(0,\infty)}(R,R^+)_E}/(\phi^*\mathscr{O}_{\mathcal{Y}_{(0,\infty)}(R,R^+)_E} \cong \mathscr{O}_{\mathcal{Y}_{(0,\infty)}(R,R^+)_E} \xrightarrow{b_S} \mathscr{O}_{\mathcal{Y}_{(0,\infty)}(R,R^+)_E})$$

where we note $b_S \in (E_0 \times_{\mathbb{Q}_p} E)^\times = S(E_0)$. Under the construction of Section 3.2.4, this corresponds to the S-torsor on $\mathcal{X}_{\mathrm{FF}}(R,R^+)$ given by Frobenius-descending the isomorphism

$$\phi^*(S \times_{\mathbb{Q}_p} \mathcal{Y}_{(0,\infty)}(R,R^+)) \cong S \times_{\mathbb{Q}_p} \mathcal{Y}_{(0,\infty)}(R,R^+) \xrightarrow{r_{b_S^{-1}}} S \times_{\mathbb{Q}_p} \mathcal{Y}_{(0,\infty)}(R,R^+),$$

which is the torsor $\mathscr{P}_{b_S}$ considered in Section 3.1.4.

3.2.6. On the other hand, for every perfectoid E-algebra $(R^\sharp, R^{\sharp+})$ with tilt (R,R^+), there is a distinguished point

$$\infty\colon \mathrm{Spa}(R^\sharp, R^{\sharp+}) \hookrightarrow \mathcal{X}_{\mathrm{FF}}(R,R^+)_E$$

that is moreover a Cartier divisor, see [SW20, Proposition 11.3.1]. We now define the line bundle

$$\mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(R,R^+)_E}(\infty) \supseteq \mathscr{O}$$

that is the sheaf of rational functions on $\mathcal{X}_{\mathrm{FF}}(R,R^+)_E$ with possibly a pole at ∞ of order at most 1. Again, this defines a map

$$\mathrm{BL}_{\mu_S^{-1}}\colon \mathrm{Spd}\,E \to \mathrm{Bun}_S,$$

where it is clear from the construction that this agrees with map described in Section 3.2.1 for the torus S and the cocharacter μ_S.

3.2.7. By [LT65] there exists, up to isomorphism, a unique 1-dimensional formal group $\mathcal{LT}_{\pi_E}$ over $\mathcal{O}_E$ with an $\mathcal{O}_E$-module structure with the property that $[\pi_E] \equiv \mathrm{Frob}_q \pmod{\pi_E}$. As in [FS24, Section II.2.1], one may choose a coordinate $\mathcal{LT}_{\pi_E} \cong \operatorname{Spf} \mathcal{O}_E[[X]]$ for which the map

$$\log\colon \mathcal{LT}_{\pi_E,E} \to \widehat{\mathbb{G}}_{a,E}; \quad X \mapsto \sum_{n=0}^{\infty} \pi_E^{-n} X^{q^n}$$

of formal schemes over E is $\mathcal{O}_E$-linear. We consider the universal cover

$$\widetilde{\mathcal{LT}}_{\pi_E} = \varprojlim_{\pi_E} \mathcal{LT}_{\pi_E} \cong \operatorname{Spf} \mathcal{O}_E[[\tilde{X}^{1/p^\infty}]], \quad \tilde{X} = \lim_{n\to\infty} X_n^{q^n}$$

where X_n is the coordinate on the nth copy of $\mathcal{LT}_{\pi_E} \cong \operatorname{Spf} \mathcal{O}_E[[X_n]]$ and q is the order of the residue field k_E of E.

Proposition 3.2.8 ([FS24, Proposition II.2.2])**.** *Let $(R^\sharp, R^{\sharp+})$ be a perfectoid E-algebra with tilt (R, R^+), which is naturally a k_E-algebra. Write*

$$c = (1, 0, \dots, 0) \in \prod_{E_0 \hookrightarrow E} E \cong E_0 \otimes_{\mathbb{Q}_p} E,$$

where 1 is at the coordinate corresponding to the natural embedding. Then the map

$$\begin{aligned} \widetilde{\mathcal{LT}}_{\pi_E}(R^{\sharp+}) \cong R^{\circ\circ} &\to H^0(Y_{(0,\infty)}(R,R^+)_E, \mathscr{O}_{Y_{(0,\infty)}(R,R^+)_E})^{\phi=b_S^{-1}} \\ &\cong H^0(\mathcal{X}_{\mathrm{FF}}(R,R^+)_E, \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(R,R^+)_E}(b_S)) \\ X &\mapsto \sum_{n=-\infty}^{\infty} b_S\phi(b_S)\cdots\phi^{n-1}(b_S)\phi^n(c)([X^{p^n}] \otimes 1) \end{aligned}$$

is an E-linear isomorphism. Under this isomorphism, the evaluation map

$$H^0(\mathcal{X}_{\mathrm{FF}}(R,R^+)_E, \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(R,R^+)_E}(b_S)) \to R^\sharp$$

at ∞ is identified with

$$\widetilde{\mathcal{LT}}_{\pi_E}(R^{\sharp+}) \to \mathcal{LT}_{\pi_E}(R^{\sharp+}) \xrightarrow{\log} R^\sharp.$$

3.2.9. We choose a nonzero π_E-torsion point $\lambda_1 \in \mathcal{LT}(\mathcal{O}_{\mathbb{C}_p}) \cong \mathfrak{m}_{\mathbb{C}_p}$, and also choose points $\lambda_2, \lambda_3, \dots \in \mathcal{LT}(\mathcal{O}_{\mathbb{C}_p}) = \mathfrak{m}_{\mathbb{C}_p}$ so that $[\pi_E](\lambda_i) = \lambda_{i-1}$. Let

$$\lambda = (0, \lambda_1, \lambda_2, \dots) \in \widetilde{\mathcal{LT}}(\mathcal{O}_{\mathbb{C}_p}).$$

For each element $u \in \mathcal{O}_E^\times$, we may define

$$[u](\lambda) = (0, [u](\lambda_1), [u](\lambda_2), \dots) \in \widetilde{\mathcal{LT}}(\mathcal{O}_{\mathbb{C}_p}),$$

noting that each $[u](\lambda_n)$ is well-defined as it only depends on u mod $\pi_E^n \mathcal{O}_E$.

Theorem 3.2.10 ([LT65, Theorem 3])**.** *We have $\lambda_1, \lambda_2, \dots \in E^{\mathrm{ab}}$, where E^{ab} is the maximal abelian extension of E. Moreover, for every $n \in \mathbb{Z}$ and $u \in \mathcal{O}_E^\times$, we have*

$$\mathrm{Art}_E(u\pi_E^n)(\lambda) = [u](\lambda)$$

where $\mathrm{Art}_E\colon E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$ is the Artin reciprocity map.

3.2.11. We consider the completed maximal abelian extension $\widehat{E}^{\mathrm{ab}}/E$, so that $\breve{E} \subseteq \widehat{E}^{\mathrm{ab}}$. Because $\widehat{E}^{\mathrm{ab}}$ is a (completed) pro-étale extension of $\mathbb{Q}_p^{\mathrm{cyc}} = \mathbb{Q}_p(\zeta_{p^\infty})^\wedge$, which is perfectoid, we see that $\widehat{E}^{\mathrm{ab}}$ is a perfectoid field as well. Using Proposition 3.2.8, we obtain a nonzero section

$$s \in H^0(\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E, \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S))$$

corresponding to $\lambda \in \widetilde{\mathcal{LT}}_{\pi_E}(\mathcal{O}_{\widehat{E}^{\mathrm{ab}}})$. On the other hand, it is clear that $\log(\lambda) = 0$, and hence $s|_{\mathrm{Spa}\,\widehat{E}^{\mathrm{ab}}} = 0$. That is, the section s defines a morphism

$$s\colon \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(\infty) \to \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S),$$

which is an isomorphism by [Far20, Proposition 2.17]. Equivalently, s defines an homotopy filling in

$$\begin{tikzcd} \mathrm{Spd}\,\widehat{E}^{\mathrm{ab}} \arrow[rr, "\rho"] \arrow[d] & & \mathrm{Spd}\,\overline{\mathbb{F}}_p \arrow[d, "b_S"] \\ \mathrm{Spd}\,E = [\mathrm{Spd}\,\widehat{E}^{\mathrm{ab}}/\underline{\mathrm{Gal}(E^{\mathrm{ab}}/E)}] \arrow[r, "\mathrm{BL}_{\mu_S^{-1}}"] & \mathrm{Bun}_S^{[b_S]} = [\mathrm{Spd}\,\overline{\mathbb{F}}_p/\underline{A_{S,b_S}}], \end{tikzcd}$$

where $\rho\colon \mathrm{Spd}\,\widehat{E}^{\mathrm{ab}} \to \mathrm{Spd}\,\overline{\mathbb{F}}_p$ is projection to its perfect residue field.

3.2.12. Taking the Čech nerve of both vertical maps, we obtain a map

$$\underline{\mathrm{Gal}(E^{\mathrm{ab}}/E)} \times \mathrm{Spd}\,\widehat{E}^{\mathrm{ab}} \xrightarrow{(\mathrm{pr}_2, \mathrm{act}), \cong} \mathrm{Spd}\,\widehat{E}^{\mathrm{ab}} \times_{\mathrm{Spd}\,E} \mathrm{Spd}\,\widehat{E}^{\mathrm{ab}}$$
$$\to \mathrm{Spd}\,\overline{\mathbb{F}}_p \times_{\mathrm{Bun}_S} \mathrm{Spd}\,\overline{\mathbb{F}}_p \xleftarrow{(\mathrm{pr}_2, \mathrm{act}), \cong} \underline{A_{S,b_S}} \times \mathrm{Spd}\,\overline{\mathbb{F}}_p.$$

This may be described as follows. Given any $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, we obtain an isomorphism

$$\gamma_\sigma\colon \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S) \xrightarrow{s^{-1}} \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(\infty)$$
$$\cong \sigma^*\mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(\infty) \xrightarrow{\sigma^* s} \sigma^*\mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S).$$

A σ-conjugated automorphism of $\mathscr{O}(b_S)$ corresponds to an element of A_{S,b_S} contained in the fiber of $\sigma|_{\overline{\mathbb{F}}_p} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, see Section 3.1.4. Thus the construction $\sigma \mapsto \gamma_\sigma$ defines a continuous homomorphism of locally profinite groups[7]

$$\alpha\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to A_{S,b_S}$$

lifting the restriction map $\mathrm{Gal}(E^{\mathrm{ab}}/E) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. It follows from the construction that the map on morphisms $\mathrm{Spd}\,\widehat{E}^{\mathrm{ab}} \times \underline{\mathrm{Gal}(E^{\mathrm{ab}}/E)} \to \mathrm{Spd}\,\overline{\mathbb{F}}_p \times \underline{A_{S,b_S}}$ is simply $\rho \times \alpha$, and hence the Beauville–Laszlo map is identified as

$$\mathrm{BL}_{\mu_S^{-1}}\colon \mathrm{Spd}\,E = [\mathrm{Spd}\,\widehat{E}^{\mathrm{ab}}/\underline{\mathrm{Gal}(E^{\mathrm{ab}}/E)}] \xrightarrow{[\rho/\alpha]} [\mathrm{Spd}\,\overline{\mathbb{F}}_p/\underline{A_{S,b_S}}] = \mathrm{Bun}_S^{[b_S]}.$$

[7]Note that the functor from locally profinite groups to v-sheaves is fully faithful, see e.g. [Sch22, Example 11.12].

3.2.13. Our goal now is to compute the continuous group homomorphism α. Recall that for our choice of b_S, we have $b_S = \mathrm{Nm}_{E/E_0}(\mu_S(\pi_E^{-1})) \in S(E_0)$. It follows from the definition in Section 3.1.1 that the preimage of $\mathrm{Gal}(\overline{\mathbb{F}}_p/k_E) \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ under $A_{S,b_S} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is

$$S(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{\mathbb{F}}_p/k_E) \subseteq A_{S,b_S}.$$

That is, $A_{S,b_S} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ naturally splits over $\mathrm{Gal}(\overline{\mathbb{F}}_p/k_E)$. The image of α is contained in this open subgroup, since the image of $\mathrm{Gal}(E^{\mathrm{ab}}/E) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is $\mathrm{Gal}(\overline{\mathbb{F}}_p/k_E)$.

Proposition 3.2.14. *Consider the composition*

$$\beta\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \xrightarrow{\mathrm{Art}_E^{-1}} \widehat{E^\times} \cong \mathcal{O}_E^\times \times \pi_E^{\widehat{\mathbb{Z}}} \xrightarrow{\pi_E \mapsto \phi^{[k_E:\mathbb{F}_p]}} \mathcal{O}_E^\times \times \mathrm{Gal}(\overline{\mathbb{F}}_p/k_E).$$

Then we have $\alpha = -\beta$ where $\mathcal{O}_E^\times \times \mathrm{Gal}(\overline{\mathbb{F}}_p/k_E) \subseteq A_{S,b_S}$ as in Section 3.2.13 upon identifying $E^\times \cong S(\mathbb{Q}_p)$.

Proof. It follows from basic properties of Art_E that $\mathrm{pr}_2 \circ \beta$ agrees with the negation of the induced map on residue fields. On the other hand, $\mathrm{pr}_2 \circ \alpha$ is by construction the induced map on residue fields, see Section 3.2.12. Hence it remains to verify that $\mathrm{pr}_1 \circ \alpha = -\mathrm{pr}_1 \circ \beta$.

Let $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ be any automorphism. It follows from the discussion of Section 3.2.12 and Section 3.2.13 that the element $\mathrm{pr}_1(\alpha(\sigma)) \in E^\times$ corresponds to the automorphism

$$\begin{aligned}\gamma_\sigma\colon \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S) &\xrightarrow{s^{-1}} \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(\infty) \cong \sigma^*\mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(\infty) \\ &\xrightarrow{\sigma^* s} \sigma^*\mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S) \cong \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S)\end{aligned}$$

where in the last isomorphism we are using that b_S is defined over k_E. The isomorphism

$$\widetilde{\mathcal{LT}}_{\pi_E}(\mathcal{O}_{\widehat{E}^{\mathrm{ab}}}) \cong H^0(\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E, \mathscr{O}_{\mathcal{X}_{\mathrm{FF}}(\widehat{E}^{\mathrm{ab},\flat}, \mathcal{O}_{\widehat{E}^{\mathrm{ab},\flat}})_E}(b_S))$$

from Proposition 3.2.8 is both $\mathrm{Gal}(\widehat{E}^{\mathrm{ab}}/E)$-equivariant and E-linear. Hence γ_σ is multiplication by

$$\sigma(s)/s = \sigma(\lambda)/\lambda = \mathrm{pr}_1(\beta(\sigma)) \in \mathcal{O}_E^\times$$

by Theorem 3.2.10, where by $\sigma(\lambda)/\lambda$ we mean the unique element $x \in \mathcal{O}_E^\times$ for which $x\lambda = \sigma(\lambda)$.

Following the constructions from Section 3.2.4 and Section 3.1.4, we see that the corresponding map $\mathscr{P}_{b_S} \to \sigma^*\mathscr{P}_{b_S}$ is right multiplication by $\mathrm{pr}_1(\beta(\sigma))^{-1}$, where the inverse comes from the fact that the construction involves dualizing the vector bundle. Therefore $\mathrm{pr}_1(\alpha(\sigma)) = \mathrm{pr}_1(\beta(\sigma))^{-1}$ by the construction in Section 3.1.4. □

3.2.15. Recall we have a homomorphism $r_\mu\colon S = \mathrm{Res}_{E/\mathbb{Q}_p}\mathbb{G}_m \to T$. This induces a map $r_\mu\colon \mathrm{Bun}_S \to \mathrm{Bun}_T$ as well as $r_\mu\colon A_{S,b_S} \to A_{T,b}$. From the previous discussion, it follows that the Beauville–Laszlo map

$$\mathrm{BL}_{\mu^{-1}}\colon \mathrm{Spd}\,E \to \mathrm{Bun}_T$$

may be identified with

$$[\rho/(r_\mu\circ\alpha)]\colon [\mathrm{Spd}\,\widehat{E}^{\mathrm{ab}}/\underline{\mathrm{Gal}(E^{\mathrm{ab}}/E)}] \to [\mathrm{Spd}\,\overline{\mathbb{F}}_p/\underline{A_{T,b}}].$$

3.3. The fiber product diagram.

3.3.1. Let T be a torus over $\mathbb{Q}$, and let $\mu\colon \mathbb{G}_{m,\mathsf{E}} \to \mathsf{T}_{\mathsf{E}}$ be a cocharacter whose field of definition is E. Then (T,μ) is a Shimura datum with reflex field E. There is a corresponding infinite level Shimura variety

$$\mathbf{Sh}(\mathsf{T},\mu)_{\mathsf{E}} = \varprojlim_{K\to\{1\}} \mathbf{Sh}_K(\mathsf{T},\mu)_{\mathsf{E}}.$$

By [Mil05, Theorem 5.28], for example, its $\bar{\mathsf{E}}$-points are given by

$$\mathsf{Sh}(\mathsf{T},\mu)_{\mathsf{E}}(\bar{\mathsf{E}}) = \mathsf{T}(\mathbb{Q})^-\backslash \mathsf{T}(\mathbb{A}_f),$$

where $\mathsf{T}(\mathbb{Q})^-$ denotes the closure of $\mathsf{T}(\mathbb{Q})$ inside $\mathsf{T}(\mathbb{A}_f)$.

3.3.2. The Galois action is defined as follows. There is an Artin isomorphism

$$\mathrm{Art}_{\mathsf{E}}\colon (\mathsf{E}^\times)^-\backslash\Big(\mathbb{A}_{\mathsf{E}}^{\infty,\times} \times \prod_{\mathsf{E}_v \xrightarrow{\sim}\mathbb{R}} (\mathbb{Z}/2\mathbb{Z})\Big) \cong \mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E})$$

with the conventions from Section 1.7. On the other hand, the cocharacter μ induces a morphism $r_\mu = \sum_{\mathsf{E}\to\bar{\mathbb{Q}}}\mu\colon \mathrm{Res}_{\mathsf{E}/\mathbb{Q}}\mathbb{G}_m \to \mathsf{T}$ of tori. Combining the two, we obtain a map

$$\mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \xrightarrow{\mathrm{Art}_{\mathsf{E}}^{-1}} (\mathsf{E}^\times)^-\backslash\mathbb{A}_{\mathsf{E}}^{\infty,\times} \xrightarrow{r_\mu} \mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{A}_f).$$

The Galois descent datum on the infinite level Shimura variety $\mathbf{Sh}(\mathsf{T},\mu)_{\mathsf{E}}$ is defined so that $\sigma\in\mathrm{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E})$ acts on $\mathbf{Sh}(\mathsf{T},\mu)_{\mathsf{E}}(\bar{\mathsf{E}}) \cong \mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{A}_f)$ as translation by $r_\mu(\mathrm{Art}_{\mathsf{E}}^{-1}(\sigma))$. This means that we may identify

$$\mathbf{Sh}(\mathsf{T},\mu)_{\mathsf{E}} \cong [(\underline{\mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{A}_f)} \times \mathrm{Spec}\,\bar{\mathsf{E}})/\,\mathrm{Gal}(\bar{\mathsf{E}}/\mathsf{E})]$$

where the right $\mathrm{Gal}(\bar{\mathsf{E}}/\mathsf{E})$-action on $\mathrm{Spec}\,\bar{\mathsf{E}}$ is the usual one and the right $\mathrm{Gal}(\bar{\mathsf{E}}/\mathsf{E})$-action on $\mathsf{T}(\mathbb{Q})^-\backslash\mathsf{T}(\mathbb{A}_f)$ is translation by $-r_\mu(\mathrm{Art}_{\mathsf{E}}^{-1})$.

3.3.3. We now fix a place $v \mid p$ of E and base change to the local field $E = \mathsf{E}_v$. We also write $T = \mathsf{T}_{\mathbb{Q}_p}$, so that using the embedding $\mathsf{E} \hookrightarrow E$ we may regard μ as a geometric cocharacter of T with field of definition E. There is a corresponding map of $\mathbb{Q}_p$-tori

$$r_{\mu,v} \colon \operatorname{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m \to T.$$

It is related to r_μ via the inclusion

$$\begin{tikzcd} \operatorname{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m \arrow[r, hook] \arrow[d, "r_{\mu,v}"] & (\operatorname{Res}_{\mathsf{E}/\mathbb{Q}} \mathbb{G}_m)_{\mathbb{Q}_p} \arrow[d, "r_\mu"] \\ T \arrow[r, equal] & \mathsf{T}_{\mathbb{Q}_p}, \end{tikzcd}$$

where on $\mathbb{Q}_p$-points the top horizontal arrow is given by $E^\times \hookrightarrow (\mathsf{E} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ corresponding to the place $v \mid p$.

3.3.4. Consider the Shimura variety base-changed to the local field E. It has the same $\bar{E}$-points

$$\mathbf{Sh}(\mathsf{T}, \mu)_E(\bar{E}) = \mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f),$$

and the Galois action is given as the restriction along $\operatorname{Gal}(E^{\mathrm{ab}}/E) \to \operatorname{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E})$. By compatibility of the local and global Artin maps, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Gal}(\mathsf{E}^{\mathrm{ab}}/\mathsf{E}) \arrow[r, "\operatorname{Art}_{\mathsf{E}}^{-1}"] & (\mathsf{E}^\times)^- \backslash \mathbb{A}_{\mathsf{E}}^{\infty,\times} \arrow[r, "r_\mu"] & \mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f) \\ \operatorname{Gal}(E^{\mathrm{ab}}/E) \arrow[u, "\mathrm{res}"] \arrow[r, "\operatorname{Art}_E^{-1}"] & \widehat{E^\times} \arrow[u, "\tilde{\iota}"] \arrow[ur] & \end{tikzcd}$$

where $\widehat{E^\times}$ is the profinite completion of $E^\times$, and $\tilde{\iota}$ is the unique extension of $\iota \colon E^\times \hookrightarrow \mathbb{A}_{\mathsf{E}}^{\infty,\times} \twoheadrightarrow (\mathsf{E}^\times)^- \backslash \mathbb{A}_{\mathsf{E}}^{\infty,\times}$ to a continuous map. It follows that the local Galois action on the Shimura variety can be described as translation by $r_\mu \circ \tilde{\iota}$, which is the unique extension of

$$E^\times \xrightarrow{r_{\mu,v}} T(\mathbb{Q}_p) \to \mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f)$$

to a continuous map.

3.3.5. As in Section 3.2.3, we choose a uniformizer $\pi_E \in E$ and let $b = \operatorname{Nm}_{E/E_0}(\mu(\pi_E^{-1})) \in T(E_0) \subseteq T(\breve{\mathbb{Q}}_p)$. We have constructed in Definition 3.1.11 the Igusa stack $\operatorname{Igs}(\mathsf{T}, b) \to \operatorname{Bun}_T^{[b]}$, and we have given in Section 3.2.15 an explicit description of the Beauville–Laszlo map $\operatorname{BL}_{\mu^{-1}} \colon \operatorname{Spd} E \to \operatorname{Bun}_T^{[b]}$. We can now compute the fiber product $\operatorname{Igs}(\mathsf{T}, b) \times_{\operatorname{Bun}_T} \operatorname{Spd} E$ as

$$\begin{tikzcd} {[(\underline{\mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f)} \times \operatorname{Spd} \widehat{E}^{\mathrm{ab}})/\underline{\operatorname{Gal}(E^{\mathrm{ab}}/E)}]} \arrow[r] \arrow[d] & {[\operatorname{Spd} \widehat{E}^{\mathrm{ab}}/\underline{\operatorname{Gal}(E^{\mathrm{ab}}/E)}]} \arrow[d, "\operatorname{BL}_{\mu^{-1}} = {[\rho/(r_{\mu,v} \circ \alpha)]}"] \\ \operatorname{Igs}(\mathsf{T}, b) = {[(\underline{\mathsf{T}(\mathbb{Q})^- \backslash \mathsf{T}(\mathbb{A}_f)} \times \operatorname{Spd} \overline{\mathbb{F}}_p)/\underline{A_{T,b}}]} \arrow[r, "{[\mathrm{pr}_2/\mathrm{id}]}"] & {[\operatorname{Spd} \overline{\mathbb{F}}_p/\underline{A_{T,b}}]} = \operatorname{Bun}_T^{[b]}, \end{tikzcd}$$

where the action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ on $\mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$ is via translation along the homomorphism

$$\mathrm{Gal}(E^{\mathrm{ab}}/E) \xrightarrow{\alpha} A_{S,b_S} \xrightarrow{r_{\mu,v}} A_{T,b} \xrightarrow{\iota_{T,b}} \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f),$$

see Definition 3.1.11.

Lemma 3.3.6. *We have*

$$-r_\mu \circ \tilde{\iota} \circ \mathrm{Art}_E^{-1} = \iota_{T,b} \circ r_{\mu,v} \circ \alpha$$

as continuous group homomorphisms $\mathrm{Gal}(E^{\mathrm{ab}}/E) \to \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f)$.

Proof. Let us write $f = [k_E : \mathbb{F}_p]$. By Proposition 3.2.14, it is enough to show that the diagram

$$\begin{tikzcd} \widehat{E^\times} \arrow[rr, "\tilde{\iota}"] \arrow[d, equal] & & (\mathsf{E}^\times)^{-}\backslash\mathbb{A}_{\mathsf{E}}^{\infty,\times} \arrow[r, "r_\mu"] & \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f) \\ \mathcal{O}_E^\times \times \pi_E^{\widehat{\mathbb{Z}}} \arrow[r, "r_{\mu,v}\times(\pi_E\mapsto\phi^f)"] & T(\mathbb{Q}_p)\times \mathrm{Gal}(\overline{\mathbb{F}}_p/k_E) \arrow[rr, hook] & & A_{T,b} \arrow[u, "\iota_{T,b}"] \end{tikzcd}$$

commutes. Since $E^\times$ is dense in its profinite completion, it suffices to check that the two compositions agree upon restricting to $E^\times$. Using the characterization in Section 3.3.3, we may instead show that the diagram

$$\begin{tikzcd} E^\times \arrow[rr, "r_{\mu,v}"] \arrow[d, equal] & & T(\mathbb{Q}_p) \arrow[r, hook, "\iota"] & \mathsf{T}(\mathbb{Q})^{-}\backslash\mathsf{T}(\mathbb{A}_f) \\ \mathcal{O}_E^\times \times \pi_E^{\mathbb{Z}} \arrow[r, "r_{\mu,v}\times(\pi_E\mapsto\phi^f)"] & T(\mathbb{Q}_p)\times \mathrm{Gal}(\overline{\mathbb{F}}_p/k_E) \arrow[rr, hook] & & A_{T,b} \arrow[u, "\iota_{T,b}"] \end{tikzcd}$$

commutes.

We now evaluate both maps on $u\pi_E^n \in E^\times$, where $n \in \mathbb{Z}$ and $u \in \mathcal{O}_E^\times$. The commutativity of the diagram is equivalent to the identity

$$\iota(r_{\mu,v}(u\pi_E^n)) = \iota_{T,b}(r_{\mu,v}(u), \phi^{fn}).$$

Recall from Lemma 3.1.9 that $\iota_{T,b}(b,\phi) = 1$ by construction. We compute

$$1 = \iota_{T,b}(b,\phi)^f = \iota_{T,b}(b\phi(b)\cdots\phi^{f-1}(b), \phi^f) = \iota_{T,b}(r_{\mu,v}(\pi_E^{-1}), \phi^f)$$

using the discussion in Section 3.2.3. It follows that

$$\iota_{T,b}(r_{\mu,v}(u), \phi^{fn}) = \iota_{T,b}(r_{\mu,v}(u\pi_E^n), 1)$$

On the other hand, we also have $\iota_{T,b}(x,1) = \iota(x)$ for all $x \in T(\mathbb{Q}_p)$ by construction, see Lemma 3.1.9. Hence

$$\iota_{T,b}(r_{\mu,v}(u\pi_E^n), 1) = \iota(r_{\mu,v}(u\pi_E^n)), \tag{3.3.1}$$

showing the desired identity. □

Theorem 3.3.7. *The Igusa stack* $\mathrm{Igs}(\mathsf{T}, b) \to \mathrm{Bun}_T^{[b]}$ *from Definition 3.1.11, together with prime-to-p Hecke action from Section 3.1.15, is an Igusa stack for the Shimura datum* (T, μ).

Proof. It follows from Section 3.3.2, Section 3.3.5 and Lemma 3.3.6 that there exists an isomorphism of v-sheaves

$$\mathbf{Sh}(\mathsf{T},\mu)_E^{\diamondsuit} \cong \mathrm{Igs}(\mathsf{T},b) \times_{\mathrm{Bun}_T} \mathrm{Spd}\, E.$$

It is clear that the isomorphism is $\underline{\mathsf{T}(\mathbb{A}_f^p)}$-equivariant. Finally, it follows from Section 3.1.16 that $\phi \times_{\phi \cong \mathrm{id}} \mathrm{id}$ on the right hand side agrees with $\mathrm{id} \times_{\mathrm{id}} \mathrm{id} = \mathrm{id}$, see Definition 2.1.5. □

Remark 3.3.8. We emphasize that there are many $\mathsf{T}(\mathbb{A}_f)$-equivariant isomorphisms $\mathbf{Sh}(\mathsf{T},\mu)_E^{\diamondsuit} \cong \mathrm{Igs}(\mathsf{T},b) \times_{\mathrm{Bun}_T} \mathrm{Spd}\, E$. In other words, the argument shows that there exists a $\mathsf{T}(\mathbb{A}_f^p)$-equivariant isomorphism of Igusa stacks $\mathrm{Igs}(\mathsf{T},\mu) \cong \mathrm{Igs}(\mathsf{T},\mu')$ for certain pairs of cocharacters μ,μ'. For Hodge type Shimura varieties, such isomorphisms are expected to be closely related to mod p isogenies between the corresponding CM abelian varieties, but this relation is not clear from the above description.

4. Igusa stacks for Shimura varieties of abelian type

The goal of this section is to show that Shimura varieties of abelian type admit Igusa stacks, which we do in Theorem 4.4.1 and Corollary 4.5.4. Recall that a Shimura datum is of *abelian type* if there exists a pair $((\mathsf{G},\mathsf{X}),\Xi)$, where (G,X) is a Shimura datum of Hodge type and $\Xi : \mathsf{G}^{\mathrm{der}} \to \mathsf{G}_2^{\mathrm{der}}$ is a central isogeny inducing an isomorphism $(\mathsf{G}^{\mathrm{ad}},\mathsf{X}^{\mathrm{ad}}) \to (\mathsf{G}_2^{\mathrm{ad}},\mathsf{X}_2^{\mathrm{ad}})$. We will refer to the pair $((\mathsf{G},\mathsf{X}),\Xi)$ as an *auxiliary Hodge-type Shimura datum.* Shimura data of abelian type are classified in [Mil05, Appendix B]; in particular all Shimura data (G,X) with G of type A,B,C are of abelian type, and Shimura data of type D are often of abelian type.

As explained in the introduction, to construct an Igusa stack for a Shimura datum of abelian type $(\mathsf{G}_2,\mathsf{X}_2)$, we find a diagram of Shimura data

$$\begin{tikzcd} & (\mathsf{G}_3,\mathsf{X}_3) \arrow[dl] \arrow[dr] & \\ (\mathsf{G}_2,\mathsf{X}_2) & & (\mathsf{G},\mathsf{X}), \end{tikzcd}$$

where (G,X) is of Hodge type, where the left arrow is an ad-isomorphism and where the right arrow induces an isomorphism of derived groups. In Section 4.1, we show that the existence of Igusa stacks "ascends" from (G,X) to $(\mathsf{G}_3,\mathsf{X}_3)$. In Section 4.3, we show that the existence of Igusa stacks "descends" from $(\mathsf{G}_3,\mathsf{X}_3) \to (\mathsf{G}_2,\mathsf{X}_2)$, under an assumption on the reflex field. To do this, we recall in Section 4.2 how the Shimura varieties for $(\mathsf{G}_2,\mathsf{X}_2)$ can be constructed out of those for $(\mathsf{G}_3,\mathsf{X}_3)$. In Section 4.4, we put everything together and prove Theorem 4.4.1. We construct Igusa stacks at finite level $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ in Section 4.5.

4.1. **Ascending Igusa stacks.** The aim of this section is to prove that the construction of Igusa stacks ascends along certain pullback diagrams of Shimura data; see Theorem 4.1.5 below. We begin with two lemmas about the potentially good reduction locus of a Shimura variety.

4.1.1. Suppose that (G,X) and $(\mathsf{G}',\mathsf{X}')$ are Shimura data of abelian type, with reflex fields E and E'. Let E'' denote the compositum of E and E'. Fix a prime v'' of E'' above p, and let v and v' be the induced primes of E and E'. Denote by E, E', and E'' the completions of E, E', and E'' at v, v', and v'', respectively.

Lemma 4.1.2. *For any neat compact open subgroups K and K' of $\mathsf{G}(\mathbb{A}_f)$ and $\mathsf{G}'(\mathbb{A}_f)$, respectively, there is a canonical isomorphism*

$$\mathbf{Sh}_{K\times K'}(\mathsf{G}\times\mathsf{G}',\mathsf{X}\times\mathsf{X}')^{\circ}_{E''}\xrightarrow{\sim}\mathbf{Sh}_{K}(\mathsf{G},\mathsf{X})^{\circ}_{E''}\times_{\mathrm{Spa}\,E''}\mathbf{Sh}_{K'}(\mathsf{G}',\mathsf{X}')^{\circ}_{E''}. \tag{4.1.1}$$

Proof. Both sides of (4.1.1) are quasicompact open subsets of $\mathbf{Sh}_{K\times K'}(\mathsf{G}\times\mathsf{G}',\mathsf{X}\times\mathsf{X}')^{\mathrm{an}}_{E''}$, so by [IM20, Lemma 3.6 (i)], it is enough to check that the two have the same classical points. To a classical point x'' of $\mathbf{Sh}_{K\times K'}(\mathsf{G}\times\mathsf{G}',\mathsf{X}\times\mathsf{X}')^{\mathrm{an}}_{E''}$ we can attach a $(\mathsf{G}^{\mathrm{ad}}\times\mathsf{G}'^{,\mathrm{ad}})(\mathbb{A}_f)$-valued Galois representation $\phi^{\mathrm{ad}}_{x''}$ (see [IM20, Definition 5.1]), and by [IM20, Corollary 5.14], the point x'' lies in $\mathbf{Sh}_{K\times K'}(\mathsf{G}\times\mathsf{G}',\mathsf{X}\times\mathsf{X}')^{\circ}$ if and only if the projection $\phi^{\mathrm{ad}}_{x'',p}$ of $\phi^{\mathrm{ad}}_{x''}$ onto $(\mathsf{G}^{\mathrm{ad}}\times\mathsf{G}'^{,\mathrm{ad}})(\mathbb{Q}_p)$ is potentially crystalline.

Fix faithful algebraic representations ξ and ξ' of G^{ad} and $\mathsf{G}'^{,\mathrm{ad}}$ on finite free $\mathbb{Q}_p$-vector spaces V and V', respectively. Then $\xi''=\xi\oplus\xi'$ is a faithful representation of $\mathsf{G}^{\mathrm{ad}}\times\mathsf{G}'^{,\mathrm{ad}}$, and by [IM20, Lemma 2.9 (ii) and (iii)], $\phi^{\mathrm{ad}}_{x'',p}$ is potentially crystalline if and only if $\xi''\circ\phi^{\mathrm{ad}}_{x'',p}$ is potentially crystalline.

Let $\kappa_{x''}$ denote the residue field of x'', and let $\overline{\kappa}_{x''}$ denote an algebraic closure of $\kappa_{x''}$. Let x and x' be the classical points of $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\mathrm{an}}_{E''}$ and $\mathbf{Sh}_{K'}(\mathsf{G}',\mathsf{X}')^{\mathrm{an}}_{E''}$ mapped to by x'' under the projection morphisms. By [IM20, Proposition 5.3], the representation $\xi''\circ\phi^{\mathrm{ad}}_{x'',p}$ decomposes as the direct sum of the restrictions of $\xi\circ\phi^{\mathrm{ad}}_{x,p}$ and $\xi'\circ\phi^{\mathrm{ad}}_{x',p}$ to $\mathrm{Gal}(\overline{\kappa}_{x''}/\kappa_{x''})$, up to conjugation by $\xi(K^{\mathrm{ad}})$ and $\xi'(K'^{,\mathrm{ad}})$. Therefore $\xi''\circ\phi^{\mathrm{ad}}_{x'',p}$ is potentially crystalline if and only if both $\xi\circ\phi^{\mathrm{ad}}_{x,p}$ and $\xi'\circ\phi^{\mathrm{ad}}_{x',p}$ are potentially crystalline. In turn, $\phi^{\mathrm{ad}}_{x'',p}$ is potentially crystalline if and only if both $\phi^{\mathrm{ad}}_{x,p}$ and $\phi^{\mathrm{ad}}_{x',p}$ are potentially crystalline, and the lemma follows. ☐

Lemma 4.1.3. *Suppose $f\colon(\mathsf{G},\mathsf{X})\to(\mathsf{H},\mathsf{Y})$ is an ad-isomorphism of Shimura data. Fix a prime v of E, and let w be the induced place of F under $\mathsf{F}\subset\mathsf{E}$, and let E and F denote the completions of E and F at w and v, respectively. Let K and L be neat compact open subgroups of $\mathsf{G}(\mathbb{A}_f)$ and $\mathsf{H}(\mathbb{A}_f)$, respectively, such that $f(\mathsf{K})\subset\mathsf{L}$. Then f induces an isomorphism*

$$\mathbf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})^{\circ}\xrightarrow{\sim}\mathbf{Sh}_{\mathsf{L}}(\mathsf{H},\mathsf{Y})^{\circ}_{E}\times_{\mathbf{Sh}_{\mathsf{L}}(\mathsf{H},\mathsf{Y})^{\mathrm{an}}_{E}}\mathbf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})^{\mathrm{an}}. \tag{4.1.2}$$

Proof. This follows by assembling various results in [IM20] as in the proof of Lemma 4.1.2. Both the source and the target of (4.1.2) are locally closed constructible subsets of $\mathbf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})^{\mathrm{an}}$, so by [IM20, Lemma 3.6 (i)], it is enough to check that the two have the same classical points. To a classical point x we can attach a $\mathsf{G}^{\mathrm{ad}}(\mathbb{A}_f)$-valued Galois representation ϕ^{ad}_{x} (see [IM20, Definition 5.1]), and by [IM20, Corollary 5.14], x is in the potentially good reduction locus of $\mathbf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})^{\mathrm{an}}$ if and only if the projection $\phi^{\mathrm{ad}}_{x,p}$ of ϕ^{ad}_{x} onto $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q}_p)$ is potentially crystalline. It is clear that this can be checked after projecting to the Shimura variety for the adjoint group, and the lemma follows. ☐

Proposition 4.1.4. *Let the notation be as in Section 4.1.1. Fix $? \in \{\emptyset, \circ\}$. Suppose (G, X) admits an Igusa stack $\mathrm{Igs}^?(\mathsf{G}, \mathsf{X})$ for ? at the place v, and $(\mathsf{G}', \mathsf{X}')$ admits an Igusa stack $\mathrm{Igs}^?(\mathsf{G}', \mathsf{X}')$ for ? at the place v'. Then the product $\mathrm{Igs}^?(\mathsf{G}, \mathsf{X}) \times \mathrm{Igs}^?(\mathsf{G}', \mathsf{X}')$ is an Igusa stack for the product Shimura datum $(\mathsf{G} \times \mathsf{G}', \mathsf{X} \times \mathsf{X}')$ for ? at the place v''.*

Proof. Let $G = \mathsf{G}_{\mathbb{Q}_p}$ and $G' = \mathsf{G}'_{\mathbb{Q}_p}$. Then we have a natural isomorphism $\mathrm{Bun}_{G \times G'} \equiv \mathrm{Bun}_G \times \mathrm{Bun}_{G'}$, and thus the proposition for the entire Shimura variety (i.e., $? = \emptyset$) will follow formally once we verify the decomposition

$$\mathrm{Gr}_{G \times G', (\mu^{-1} \times \mu'^{-1})} \cong (\mathrm{Gr}_{G, \mu^{-1}})_{E''} \times_{\mathrm{Spd}\, E''} (\mathrm{Gr}_{G', \mu'^{-1}})_{E''}.$$

There is a natural map

$$\mathrm{Gr}_{G \times G', (\mu^{-1} \times \mu'^{-1})} \to (\mathrm{Gr}_{G, \mu^{-1}})_{E''} \times_{\mathrm{Spd}\, E''} (\mathrm{Gr}_{G', \mu'^{-1}})_{E''} \tag{4.1.3}$$

induced by the projections $G \times G' \to G$ and $G \times G' \to G'$. By [SW20, Proposition 20.2.3] and [Sch22, Corollary 11.29], both $\mathrm{Gr}_{G \times G', (\mu^{-1} \times \mu'^{-1})}$ and $(\mathrm{Gr}_{G, \mu^{-1}})_{E''} \times_{\mathrm{Spd}\, E''} (\mathrm{Gr}_{G', \mu'^{-1}})_{E''}$ are spatial diamonds, and therefore the map (4.1.3) is qcqs. By [Sch22, Lemma 12.5] it is then enough to check that (4.1.3) is a bijection on $\mathrm{Spa}(C, C^+)$-points for all algebraically closed perfectoid fields (C, C^+) of characteristic p, and this follows from the definition of Schubert cells in the affine Grassmannian [SW20, Definition 19.2.2].

For the potentially crystalline locus (i.e., $? = \circ$), we observe that by Lemma 4.1.2, for any pair of neat compact open subgroups K and K' of $\mathsf{G}(\mathbb{A}_f)$ and $\mathsf{G}'(\mathbb{A}_f)$ we have the decomposition (4.1.1). Since any compact open subgroup of $(\mathsf{G} \times \mathsf{G}')(\mathbb{A}_f)$ is contained in one of the form $K \times K'$ for such a K and K', by taking limits of both sides of (4.1.1) we obtain

$$\mathbf{Sh}(\mathsf{G} \times \mathsf{G}', \mathsf{X} \times \mathsf{X}')^\circ \cong \mathbf{Sh}(\mathsf{G}, \mathsf{X})^\circ_{E''} \times_{\mathrm{Spa}\, E''} \mathbf{Sh}(\mathsf{G}', \mathsf{X}')^\circ_{E''},$$

and once again the result follows formally. □

Now we can prove the main result of this section.

Theorem 4.1.5. *Fix $? \in \{\emptyset, \circ\}$. Let (G, X) and (H, Y) be Shimura data of abelian type with reflex fields E and F, and let*

$$f \colon (\mathsf{G}, \mathsf{X}) \to (\mathsf{H}, \mathsf{Y})$$

be a morphism of Shimura data which induces an isomorphism on derived groups. Fix a prime v of E above a rational prime p, let w be the induced prime of F under $\mathsf{F} \subset \mathsf{E}$, and let E and F be the corresponding completions. If (H, Y) admits an Igusa stack $\mathrm{Igs}^?(\mathsf{H}, \mathsf{Y})$ for ? at the place w, then (G, X) admits an Igusa stack $\mathrm{Igs}^?(\mathsf{G}, \mathsf{X})$ at the place v.

Proof. Let $\mathsf{E}' \subset \mathsf{E}$ denote the reflex field of $(\mathsf{G}^{\mathrm{ab}}, \mathsf{X}^{\mathrm{ab}})$, and let v' be the prime of E' induced by v. The product $(\mathsf{H} \times \mathsf{G}^{\mathrm{ab}}, \mathsf{Y} \times \mathsf{X}^{\mathrm{ab}})$ is a Shimura datum with reflex field given by the compositum $\mathsf{E}'' \subset \mathsf{E}$ of F and E'. By Theorem 3.3.7, $(\mathsf{G}^{\mathrm{ab}}, \mathsf{X}^{\mathrm{ab}})$ admits an Igusa stack for ? at v', so by Proposition 4.1.4, $(\mathsf{H} \times \mathsf{G}^{\mathrm{ab}}, \mathsf{Y} \times \mathsf{X}^{\mathrm{ab}})$ admits an

Igusa stack $\mathrm{Igs}^?(\mathsf{H}\times\mathsf{G}^{\mathrm{ab}},\mathsf{Y}\times\mathsf{X}^{\mathrm{ab}})$ for ? at the prime v'' of E'' induced by v. Since $f\colon \mathsf{G}\to\mathsf{H}$ induces an isomorphism on derived groups, the diagram

$$\begin{tikzcd} (\mathsf{G},\mathsf{X}) \arrow[r] \arrow[d] & (\mathsf{G}^{\mathrm{ab}},\mathsf{X}^{\mathrm{ab}}) \arrow[d] \\ (\mathsf{H},\mathsf{Y}) \arrow[r] & (\mathsf{H}^{\mathrm{ab}},\mathsf{Y}^{\mathrm{ab}}) \end{tikzcd} \tag{4.1.4}$$

is Cartesian. Thus we have a closed embedding of Shimura data

$$(\mathsf{G},\mathsf{X})\hookrightarrow(\mathsf{H}\times\mathsf{G}^{\mathrm{ab}},\mathsf{Y}\times\mathsf{X}^{\mathrm{ab}}),$$

and we conclude using [Kim25, Theorem 11.4], which shows that the existence of Igusa stacks passes to sub Shimura data, together with Lemma 4.1.3. ☐

4.2. **Descending Shimura varieties.** The goal of this section is to prove Proposition 4.2.6, which relates two Shimura varieties along an ad-isomorphism of Shimura data.

4.2.1. Let (G,X) be a Shimura datum and let Z_{G} be the center of G. Let $\mathsf{G}^{\mathrm{ad}}(\mathbb{R})^1$ be the image of $\mathsf{G}(\mathbb{R})\to\mathsf{G}^{\mathrm{ad}}(\mathbb{R})$ and let $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ be its intersection with $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})$. Note that $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})$ acts on G by conjugation and that $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ acts on (G,X) by morphisms of Shimura data.

4.2.2. For $K\subset\mathsf{G}(\mathbb{A}_f)$ a neat compact open subgroup we will write $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ for the Shimura variety over E of level K, and we will write $\mathbf{Sh}(\mathsf{G},\mathsf{X})=\varprojlim_K\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ for the inverse limit. This inverse limit comes equipped with a continuous left action of the locally profinite group $\mathsf{G}(\mathbb{A}_f)$, or equivalently, a left action of the locally profinite group scheme $\underline{\mathsf{G}(\mathbb{A}_f)}$ over E. We will write $Z_{\mathsf{G}}(\mathbb{Q})^-\subset\mathsf{G}(\mathbb{A}_f)$ for the closure of $Z_{\mathsf{G}}(\mathbb{Q})$ inside $\mathsf{G}(\mathbb{A}_f)$. The following lemma is well known.

Lemma 4.2.3. *The action of $\underline{Z_{\mathsf{G}}(\mathbb{Q})^-}\subset\underline{\mathsf{G}(\mathbb{A}_f)}$ on $\mathbf{Sh}(\mathsf{G},\mathsf{X})$ is trivial.*

Proof. Since the inverse limit $\mathbf{Sh}(\mathsf{G},\mathsf{X})$ is separated scheme, the locus

$$Z=\{(g,x)\in\underline{Z_{\mathsf{G}}(\mathbb{Q})^-}\times\mathbf{Sh}(\mathsf{G},\mathsf{X}) : gx=x\}$$

is a closed subscheme. By [Del79, Corollary 2.1.12], the locus Z contains all $\mathbb{C}$-points, and thus $Z=\underline{Z_{\mathsf{G}}(\mathbb{Q})^-}\times\mathbf{Sh}(\mathsf{G},\mathsf{X})$. ☐

4.2.4. By considering the long exact sequences in cohomology with $\mathbb{Q}$ and $\mathbb{R}$ coefficients associated with $1\to Z_{\mathsf{G}}\to\mathsf{G}\to\mathsf{G}^{\mathrm{ad}}\to1$, we see that the subgroup $\mathsf{G}(\mathbb{Q})/Z_{\mathsf{G}}(\mathbb{Q})\subset\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ is normal with abelian cokernel given by $\ker(H^1(\mathbb{Q},Z_{\mathsf{G}})\to H^1(\mathbb{R},Z_{\mathsf{G}})\times H^1(\mathbb{Q},\mathsf{G}))$.[8] Together with Lemma 4.2.3, this gives rise to an action of the abstract group[9]

$$\mathcal{A}(\mathsf{G}):=\left(\frac{\mathsf{G}(\mathbb{A}_f)}{Z_{\mathsf{G}}(\mathbb{Q})^-}\rtimes\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1\right)\Big/\frac{\mathsf{G}(\mathbb{Q})}{Z_{\mathsf{G}}(\mathbb{Q})}$$

[8] In particular, this cokernel is finite if Z_{G} is connected.

[9] The multiplication in the semidirect product is given by $(x,g)(y,h)=(x\cdot{}^gy,gh)$ and the map $\mathsf{G}(\mathbb{Q})\to(\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-)\rtimes\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ is given by $g\mapsto(g^{-1},g)$, cf. [Kis10, Section 3.3.1].

on $\mathbf{Sh}(\mathsf{G},\mathsf{X})$. We now equip $\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-$ with the quotient topology, the groups $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ and $\mathsf{G}(\mathbb{Q})/Z_{\mathsf{G}}(\mathbb{Q})$ with the discrete topology, the product $(\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-)\rtimes \mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ with the product topology, and $\mathcal{A}(\mathsf{G})$ with the quotient topology; note that this is Hausdorff since $\mathsf{G}(\mathbb{Q})/Z_{\mathsf{G}}(\mathbb{Q})$ has closed image. It follows that $\underline{\mathcal{A}(\mathsf{G})}$ acts on $\mathbf{Sh}(\mathsf{G},\mathsf{X})$ since $\underline{\mathsf{G}(\mathbb{A}_f)}$ and $\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1$ both act on $\mathbf{Sh}(\mathsf{G},\mathsf{X})$.

4.2.5. If $f:(\mathsf{G},\mathsf{X})\to(\mathsf{H},\mathsf{Y})$ is a morphism of Shimura data such that $f(Z_{\mathsf{G}})\subset Z_{\mathsf{H}}$, then there is an induced morphism $\mathcal{A}(\mathsf{G})\to\mathcal{A}(\mathsf{H})$ and the corresponding morphism of Shimura varieties $\mathbf{Sh}(\mathsf{G},\mathsf{X})\to\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$ is $\underline{\mathcal{A}(\mathsf{G})}$-equivariant in the obvious way. The following result is well known (see e.g. [YZ25, Lemma 4.56], [Del79, Lemma 2.7.11.(b)]). We give a proof for the sake of completeness, following [Del79, Section 2].

Proposition 4.2.6. *If $f:(\mathsf{G},\mathsf{X})\to(\mathsf{H},\mathsf{Y})$ is an ad-isomorphism, then the natural map $\mathbf{Sh}(\mathsf{G},\mathsf{X})\to\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$ induces an isomorphism*

$$\underline{\mathcal{A}(\mathsf{H})}\times^{\underline{\mathcal{A}(\mathsf{G})}}\mathbf{Sh}(\mathsf{G},\mathsf{X})\to\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}},$$

where the quotient is taken in the pro-étale topology.

Proof. There is indeed a natural map of E-schemes

$$\pi\colon \underline{\mathcal{A}(\mathsf{H})}\times\mathbf{Sh}(\mathsf{G},\mathsf{X})\to\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$$

that is invariant for the $\underline{\mathcal{A}(\mathsf{G})}$-action on the source. It remains to show that π is a pro-étale $\underline{\mathcal{A}(\mathsf{G})}$-torsor.

It suffices to show that π is weakly étale, an fpqc cover, and a quasi-torsor for $\underline{\mathcal{A}(\mathsf{G})}$ in the sense that the action map

$$\alpha\colon(\underline{\mathcal{A}(\mathsf{H})}\times\mathbf{Sh}(\mathsf{G},\mathsf{X}))\times\underline{\mathcal{A}(\mathsf{G})}\to(\underline{\mathcal{A}(\mathsf{H})}\times\mathbf{Sh}(\mathsf{G},\mathsf{X}))\times_{\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}}(\underline{\mathcal{A}(\mathsf{H})}\times\mathbf{Sh}(\mathsf{G},\mathsf{X}))$$

is an isomorphism. Note that once we fix a neat level $K\subseteq\mathsf{H}(\mathbb{A}_f)$, both sides of π are disjoint unions of pro-finite étale covers of $\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$, and hence weakly étale. It then follows from [BS15, Proposition 2.2.3.(4)] that π is weakly étale as well. To show that π is an fpqc cover, it suffices to find a compact open subset $U\subseteq\mathcal{A}(\mathsf{H})$ for which $\underline{U}\times\mathbf{Sh}(\mathsf{G},\mathsf{X})\to\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$ is surjective. This follows from the fact that a connected component of $\mathbf{Sh}(\mathsf{G},\mathsf{X})$ surjects onto a connected component of $\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$, and that $\pi_0(\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathbb{C}})=\mathsf{H}(\mathbb{A}_f)/\mathsf{H}(\mathbb{Q})_+^-$ as profinite sets, see [Del79, Proposition 2.1.14].[10] Indeed, because $\mathsf{H}(\mathbb{A}_f)/\mathsf{H}(\mathbb{Q})_+^-$ is profinite, there exists a compact open $\tilde{U}\subseteq\mathsf{H}(\mathbb{A}_f)$ that surjects onto $\mathsf{H}(\mathbb{A}_f)/\mathsf{H}(\mathbb{Q})_+^-$ as in Proposition 2.2.2, and then an open neighborhood U of $\mathrm{im}(\tilde{U}\to\mathcal{A}(\mathsf{H}))$ has the property that $\underline{U}\times\mathbf{Sh}(\mathsf{G},\mathsf{X})\to\mathbf{Sh}(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$ is surjective.

We now prove that α is an isomorphism. Recall that both sides are disjoint unions of pro-finite étale covers over $\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$. If we write $\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathsf{E}}=\coprod_i V_i$ where V_i are connected, then the category of schemes that are a disjoint union of pro-finite étale covers of V_i is equivalent to the category of locally profinite sets

[10] Here we write $\mathsf{G}(\mathbb{R})_+$ for the inverse image of the identity component (in the real topology) of $\mathsf{G}^{\mathrm{ad}}(\mathbb{R})$ under the natural map $\mathsf{G}(\mathbb{R})\to\mathsf{G}^{\mathrm{ad}}(\mathbb{R})$, and $\mathsf{G}(\mathbb{Q})_+=\mathsf{G}(\mathbb{Q})\cap\mathsf{G}(\mathbb{R})_+$.

with a continuous $\pi_1^{\text{ét}}(V_i, \bar{v}_i)$-action. Thus it is enough to show that α induces isomorphisms on fibers over $\mathbb{C}$-points of $\mathbf{Sh}_K(\mathsf{H},\mathsf{Y})_{\mathsf{E}}$. Since these fibers correspond to locally profinite sets, we may further reduce to showing that the continuous map

$$\mathcal{A}(\mathsf{H}) \times \mathbf{Sh}(\mathsf{G},\mathsf{X})(\mathbb{C})^{\text{an}} \to \mathbf{Sh}(\mathsf{H},\mathsf{Y})(\mathbb{C})^{\text{an}}$$

of topological spaces is an $\mathcal{A}(\mathsf{G})$-torsor (i.e., a principal bundle), where we give $\mathbf{Sh}(\mathsf{G},\mathsf{X})(\mathbb{C})$ and $\mathbf{Sh}(\mathsf{H},\mathsf{Y})(\mathbb{C})$ the inverse limit topologies of the analytic topologies at finite level.

In view of [Del79, Section 2.1.9], we have

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})(\mathbb{C})^{\text{an}} = \left(\frac{\mathsf{G}(\mathbb{A}_f)}{Z_{\mathsf{G}}(\mathbb{Q})^-} \times \mathsf{X}\right) \Big/ \frac{\mathsf{G}(\mathbb{Q})}{Z_{\mathsf{G}}(\mathbb{Q})}$$

as topological spaces, where the quotient is properly discontinuous. We need to check

$$S_1 = \mathcal{A}(\mathsf{H}) \times \frac{(\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-) \times \mathsf{X}}{\mathsf{G}(\mathbb{Q})/Z_{\mathsf{G}}(\mathbb{Q})} \to \frac{(\mathsf{H}(\mathbb{A}_f)/Z_{\mathsf{H}}(\mathbb{Q})^-) \times \mathsf{Y}}{\mathsf{H}(\mathbb{Q})/Z_{\mathsf{H}}(\mathbb{Q})}$$

is a torsor for $\mathcal{A}(\mathsf{G})$. It suffices to check that

$$S_2 = \mathcal{A}(\mathsf{H}) \times ((\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-) \times \mathsf{X}) \to \frac{(\mathsf{H}(\mathbb{A}_f)/Z_{\mathsf{H}}(\mathbb{Q})^-) \times \mathsf{Y}}{\mathsf{H}(\mathbb{Q})/Z_{\mathsf{H}}(\mathbb{Q})}$$

is a torsor for $\Gamma = (\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-) \rtimes \mathsf{G}^{\text{ad}}(\mathbb{Q})^1$ with the source having the diagonal action, because $S_2 \to S_1$ is a Galois covering space for the induced $\mathsf{G}(\mathbb{Q})/Z_{\mathsf{G}}(\mathbb{Q})$-action. Because being a torsor can be checked after base changing by a surjective covering map, it remains to show that

$$((\mathsf{H}(\mathbb{A}_f)/Z_{\mathsf{H}}(\mathbb{Q})^-) \rtimes \mathsf{H}^{\text{ad}}(\mathbb{Q})^1) \times ((\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-) \times \mathsf{X}) \to (\mathsf{H}(\mathbb{A}_f)/Z_{\mathsf{H}}(\mathbb{Q})^-) \times \mathsf{Y}$$

is a Γ-torsor. On the other hand, this map is $(\mathsf{H}(\mathbb{A}_f)/Z_{\mathsf{H}}(\mathbb{Q})^-)$-equivariant, hence is a base change of

$$\mathsf{H}^{\text{ad}}(\mathbb{Q})^1 \times ((\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-) \times \mathsf{X}) \to \mathsf{Y},$$

and so it suffices to show that this is a Γ-torsor. As the map is evidently Γ-invariant, this is equivalent to showing that

$$\mathsf{H}^{\text{ad}}(\mathbb{Q})^1 \times \mathsf{X} \to \mathsf{Y}$$

is a Galois covering space for $\mathsf{G}^{\text{ad}}(\mathbb{Q})^1$. This is now clear since $\mathsf{X} \hookrightarrow \mathsf{Y}$ is a union of connected components, $\mathsf{G}^{\text{ad}}(\mathbb{Q})^1 \hookrightarrow \mathsf{H}^{\text{ad}}(\mathbb{Q})^1$ is a subgroup, and

$$\mathsf{H}^{\text{ad}}(\mathbb{Q})^1 \times^{\mathsf{G}^{\text{ad}}(\mathbb{Q})^1} \pi_0(\mathsf{X}) = \mathsf{H}^{\text{ad}}(\mathbb{R})^1 \times^{\mathsf{G}^{\text{ad}}(\mathbb{R})^1} \pi_0(\mathsf{X}) = \pi_0(\mathsf{Y})$$

because X, Y are orbits for $\mathsf{G}(\mathbb{R}), \mathsf{H}(\mathbb{R})$. $\square$

4.2.7. Let $(\mathsf{G},\mathsf{X}) \to (\mathsf{H},\mathsf{Y})$ and $\mathsf{F} \subset \mathsf{E}$ be as above. For a finite place v of E with induced place w of F, we consider the completions $F \subset E$.

Corollary 4.2.8. *If $(\mathsf{G},\mathsf{X}) \to (\mathsf{H},\mathsf{Y})$ is an ad-isomorphism, then the natural map of Proposition 4.2.6 induces an isomorphism (where the quotient is taken in the pro-étale topology)*

$$\underline{\mathcal{A}(\mathsf{H})} \times^{\underline{\mathcal{A}(\mathsf{G})}} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{\circ,\diamondsuit} \to \mathbf{Sh}(\mathsf{H},\mathsf{Y})_E^{\circ,\diamondsuit}$$

Proof. This is a direct consequence of Lemma 4.1.3 and Proposition 4.2.6. $\square$

4.3. **Descending Igusa stacks.** The goal of this section is to prove the following result. Let $(\mathsf{G},\mathsf{X}) \to (\mathsf{H},\mathsf{Y})$ be a morphism of abelian-type Shimura data which induces an inclusion $\mathsf{F} \subset \mathsf{E}$ of reflex fields. Fix a prime v of E above a rational prime p, let w be the induced prime of F, and let F and E be the corresponding completions. Write $G = \mathsf{G} \otimes \mathbb{Q}_p$ and $H = \mathsf{H} \otimes \mathbb{Q}_p$.

Theorem 4.3.1. *Assume that $F = E$. If (G,X) admits an Igusa stack for $? \in \{\circ, \emptyset\}$ at the place v and $\mathsf{G} \to \mathsf{H}$ is an ad-isomorphism, then (H,Y) admits an Igusa stack for $?$ at the place w.*

The idea of the proof is relatively simple: We note that $\underline{\mathcal{A}(\mathsf{G})}$ acts on $\mathrm{Igs}^?(\mathsf{G},\mathsf{X})$ via its quotient $\underline{\mathcal{A}^p(\mathsf{G})} := \underline{\mathcal{A}(\mathsf{G})}/\underline{G(\mathbb{Q}_p)}$, and the Igusa stacks for (H,Y) should be the quotient

$$\mathrm{Igs}^?(\mathsf{H},\mathsf{Y}) := \underline{\mathcal{A}^p(\mathsf{H})} \times^{\underline{\mathcal{A}^p(\mathsf{G})}} \mathrm{Igs}^?(\mathsf{G},\mathsf{X}).$$

From this perspective, it is however not easy to see that $\mathrm{Igs}^?(\mathsf{H},\mathsf{Y})$ admits a natural map to Bun_H. Moreover, it is perhaps not clear how to make sense of the quotient if $\mathrm{Igs}^?(\mathsf{G},\mathsf{X})$ is not 0-truncated. Therefore, in the actual proof we will work from the perspective of the uniformization maps introduced by Kim in [Kim25, Section 5], see Definition 2.1.6.

4.3.2. *Local preparations.* We will need the following result in our proof of Theorem 4.3.1. Let $f : G \to H$ be an ad-isomorphism of connected reductive groups over $\mathbb{Q}_p$, and let μ be a minuscule $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters with μ_H the induced $H(\overline{\mathbb{Q}}_p)$-conjugacy class. There is an induced map $B(G,\mu) \to B(H,\mu_H)$ which is a bijection by the discussion in [Kot97, Section 6.5]. We have the following slight generalization of [Kim25, Lemma 6.4].

Lemma 4.3.3. *Suppose that $G \to H$ is an ad-isomorphism. If μ and μ_H have the same reflex field, then the 2-commutative diagram*

$$\begin{tikzcd} {[\underline{G(\mathbb{Q}_p)} \backslash \mathrm{Gr}_{G,\mu}]} \arrow[r] \arrow[d] & {[\underline{H(\mathbb{Q}_p)} \backslash \mathrm{Gr}_{H,\mu_H}]} \arrow[d] \\ \mathrm{Bun}_{G,\mu} \arrow[r] & \mathrm{Bun}_{H,\mu_H} \end{tikzcd} \tag{4.3.1}$$

is 2-Cartesian.

Proof. The proof of [Kim25, Lemma 6.4] works almost verbatim. To show that $[b]$ is trivial as in the last paragraph of the proof, we can instead argue that the Newton cocharacter ν_b has trivial image in H and thus in G^{ad} since $G \to H$ is an ad-isomorphism. The rest of the argument works to show that $[b]$ is trivial. □

4.3.4. *Local preparations II.* There is a left action of G^{ad} on the algebraic group G given by conjugation, and hence also a left action of $G^{\mathrm{ad}}(\mathbb{Q}_p)$ on $G(\mathbb{Q}_p)$. For $i = 2, 3$ we consider the locally profinite group and the closed normal subgroup

$$G(\mathbb{Q}_p)^i \rtimes G^{\mathrm{ad}}(\mathbb{Q}_p) \supseteq \{(\Delta^i(g), g^{-1}) : g \in G(\mathbb{Q}_p)\},$$

where $G^{\mathrm{ad}}(\mathbb{Q}_p)$ acts on $G(\mathbb{Q}_p)^i$ diagonally and where $\Delta^i\colon G(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^i$ is the diagonal inclusion. We see that the quotient fits in a short exact sequence

$$1 \to G(\mathbb{Q}_p)^{i-1} \xrightarrow{\underline{y} \mapsto ((1,\underline{y}),1)} \frac{G(\mathbb{Q}_p)^i \rtimes G^{\mathrm{ad}}(\mathbb{Q}_p)}{\{(\Delta^i(g), g^{-1}) : g \in G(\mathbb{Q}_p)\}} \xrightarrow{((x,\underline{y}),z) \mapsto xz} G^{\mathrm{ad}}(\mathbb{Q}_p) \to 1$$

of locally profinite groups. We will denote the locally profinite group in the middle of the above short exact sequence by $\Sigma^i(G)$.

Lemma 4.3.5. *Consider the natural $\underline{G(\mathbb{Q}_p)}^2$-action on $\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G$ as well as the $\underline{G^{\mathrm{ad}}(\mathbb{Q}_p)}$-action on $\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G$ induced by the conjugation action on G. The two actions combine into an action of $\Sigma^2(G)$ on $\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G$. Similarly, consider the natural $\underline{G(\mathbb{Q}_p)}^3$-action on $\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G$ as well as the $\underline{G^{\mathrm{ad}}(\mathbb{Q}_p)}$-action on $\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G$ induced by conjugation action on G. The two actions combine into an action of $\Sigma^3(G)$ on $\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G$.*

Proof. We first combine the two actions into an action of $\underline{G(\mathbb{Q}_p)^2 \rtimes G^{\mathrm{ad}}(\mathbb{Q}_p)}$. To show this, we need to check that first acting by $(g, h) \in \underline{G(\mathbb{Q}_p)}^2$ and then acting by $\gamma \in \underline{G^{\mathrm{ad}}(\mathbb{Q}_p)}$ is the same as first acting by γ and then by $({}^\gamma g, {}^\gamma h) \in \underline{G(\mathbb{Q}_p)}^2$. Given a triple (x, y, α) where $x \in \mathrm{Gr}_G(R^{\sharp_1}, R^{\sharp_1+})$, $y \in \mathrm{Gr}_G(R^{\sharp_2}, R^{\sharp_2+})$ and $\alpha\colon \mathscr{P}_x \cong \mathscr{P}_y$ is an isomorphism of G-torsors on $\mathcal{X}_{\mathrm{FF}}(R, R^+)$, acting by (g, h) yields the triple $(gx, hy, r_{h^{-1}}\alpha r_g)$. Then acting by γ yields $({}^\gamma g^\gamma x, {}^\gamma h^\gamma y, r_{{}^\gamma h^{-1}}{}^\gamma\alpha r_{{}^\gamma g})$. This is the same as first acting by γ and then acting by $({}^\gamma g, {}^\gamma h)$.

We next show that the subgroup $\{((g^{-1}, g^{-1}), g)\}$ acts trivially. We first note that for $g \in \underline{G(\mathbb{Q}_p)}(R, R^+)$ and $\mathscr{P}$ a G-torsor on $\mathcal{X}_{\mathrm{FF}}(R, R^+)$, there is a canonical isomorphism

$$\tau_g \colon {}^g\mathscr{P} = G \times^{\mathrm{conj}_g, G} \mathscr{P} \xrightarrow{\cong} \mathscr{P}; \quad (x, p) \mapsto xgp$$

of G-torsors. We now consider a point $(x, y, \alpha) \in (\mathrm{Gr}_G \times_{\mathrm{Bun}_G} \mathrm{Gr}_G)(R, R^+)$ as above. By naturality of τ, there is a commutative diagram

$$\begin{tikzcd}
G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)} \arrow[r, dashed, "x"] & \mathscr{P}_x \arrow[r, "\alpha"] & \mathscr{P}_y & G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)} \arrow[l, dashed, "y"'] \\
G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)} \arrow[u, "\tau_g"] \arrow[r, dashed, "{}^g x"] & {}^g\mathscr{P}_x \arrow[u, "\tau_g"] \arrow[r, "{}^g\alpha"] & {}^g\mathscr{P}_y \arrow[u, "\tau_g"] & G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)}. \arrow[u, "\tau_g"] \arrow[l, dashed, "{}^g y"']
\end{tikzcd}$$

On the other hand, we see from the formula that τ_g on the trivial G-torsor $G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)}$ agrees with right multiplication r_g by g. Because the composition

$$G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)} \xrightarrow{r_{g^{-1}}} G|_{\mathcal{X}_{\mathrm{FF}}(R,R^+)} \overset{x}{\dashrightarrow} \mathscr{P}_x$$

recovers $g^{-1}x \in \mathrm{Gr}_G(R^{\sharp_1}, R^{\sharp_1+})$, the action of $((g^{-1}, g^{-1}), g)$ on (x, y, α) recovers (x, y, α).

The statement for $\Sigma^3(G)$ follows from a similar argument. $\square$

4.3.6. Recall that there is an action of $\underline{\mathcal{A}(\mathsf{G})}$ on the v-sheaf $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}$. There is a homomorphism of locally profinite groups

$$\mathcal{A}(\mathsf{G}) = \left(\frac{\mathsf{G}(\mathbb{A}_f)}{Z_{\mathsf{G}}(\mathbb{Q})^-} \rtimes \mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1\right) \bigg/ \frac{\mathsf{G}(\mathbb{Q})}{Z_{\mathsf{G}}(\mathbb{Q})} \to G^{\mathrm{ad}}(\mathbb{Q}_p); \quad (x, y) \mapsto xy.$$

We observe that the Hodge–Tate period map

$$\pi_{\mathrm{HT}} \colon \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \to \mathrm{Gr}_{G,\mu^{-1}}$$

is equivariant with respect to this group homomorphism. It follows that the v-sheaf

$$\begin{aligned} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(2)} &:= \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \\ &= \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Gr}_{G,\mu^{-1}}} (\mathrm{Gr}_{G,\mu^{-1}} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}}) \end{aligned}$$

has an action of the locally profinite group

$$\Xi^2(\mathsf{G}) = \mathcal{A}(\mathsf{G}) \times_{G^{\mathrm{ad}}(\mathbb{Q}_p)} \Sigma^2(G).$$

This group fits in a short exact sequence

$$1 \to G(\mathbb{Q}_p) \xrightarrow{\mathrm{incl}_2} \Xi^2(\mathsf{G}) \xrightarrow{\mathrm{pr}_1} \mathcal{A}(\mathsf{G}) \to 1$$

of locally profinite groups. Similarly, the v-sheaf

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(3)} := \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}}$$

has an action of the locally profinite group $\Xi^3(\mathsf{G})$ defined as

$$1 \to G(\mathbb{Q}_p)^2 \xrightarrow{\mathrm{incl}_2} \Xi^3(\mathsf{G}) = \mathcal{A}(\mathsf{G}) \times_{G^{\mathrm{ad}}(\mathbb{Q}_p)} \Sigma^3(G) \xrightarrow{\mathrm{pr}_1} \mathcal{A}(\mathsf{G}) \to 1.$$

Lemma 4.3.7. *For $i = 2$ or $i = 3$, consider the natural map*

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(i)} \to \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(i)}$$

which is equivariant with respect to the natural group homomorphism $\Xi^i(\mathsf{G}) \to \Xi^i(\mathsf{H})$ by construction. The induced map

$$[\underline{\Xi^i(\mathsf{G})} \backslash \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(i)}] \to [\underline{\Xi^i(\mathsf{H})} \backslash \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(i)}]$$

is an isomorphism of v-stacks. In other words, there exists an $\underline{\Xi^i(\mathsf{H})}$-equivariant isomorphism

$$\underline{\Xi^i(\mathsf{H})} \times^{\underline{\Xi^i(\mathsf{G})}} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(i)} \xrightarrow{\cong} \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(i)}$$

Proof. Write $\mathrm{Gr}_{G,\mu^{-1},G(\mathbb{Q}_p)} = [\underline{G(\mathbb{Q}_p)}\backslash \mathrm{Gr}_{G,\mu^{-1}}]$ and similarly for H. Because $G(\mathbb{Q}_p) \subseteq \Xi^2(\mathsf{G})$ is the subgroup consisting of elements of the form $((1,1),((1,y),1))$, we naturally identify

$$[\underline{G(\mathbb{Q}_p)}\backslash(\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}})] = \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1},G(\mathbb{Q}_p)},$$

and moreover this v-sheaf has an induced $\underline{\mathcal{A}(\mathsf{G})}$-action. By Lemma 4.3.3 the natural diagram

$$\begin{array}{ccc} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1},G(\mathbb{Q}_p)} & \xrightarrow{\mathrm{pr}_1} & \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \\ \downarrow & & \downarrow \\ \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \times_{\mathrm{Bun}_H} \mathrm{Gr}_{H,\mu_H^{-1},H(\mathbb{Q}_p)} & \xrightarrow{\mathrm{pr}_1} & \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \end{array}$$

is Cartesian. It is also clear from the construction that this is equivariant with respect to the actions of $\mathcal{A}(\mathsf{G}) \to \mathcal{A}(\mathsf{H})$. It then formally follows that

$$\begin{array}{ccc} [\underline{\mathcal{A}(\mathsf{G})}\backslash(\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1},G(\mathbb{Q}_p)})] & \xrightarrow{\mathrm{pr}_1} & [\underline{\mathcal{A}(\mathsf{G})}\backslash\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}] \\ \downarrow & & \downarrow \\ [\underline{\mathcal{A}(\mathsf{H})}\backslash(\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \times_{\mathrm{Bun}_H} \mathrm{Gr}_{H,\mu_H^{-1},H(\mathbb{Q}_p)})] & \xrightarrow{\mathrm{pr}_1} & [\underline{\mathcal{A}(\mathsf{H})}\backslash\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}] \end{array}$$

is also Cartesian. Because the right vertical map is an isomorphism by Proposition 4.2.6 and Corollary 4.2.8, so is the left vertical map. On the other hand, it follows from the short exact sequence

$$1 \to G(\mathbb{Q}_p) \to \Xi^2(\mathsf{G}) \to \mathcal{A}(\mathsf{G}) \to 1$$

that the left vertical map can be identified with

$$[\Xi^2(\mathsf{G})\backslash\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(2)}] \to [\Xi^2(\mathsf{H})\backslash\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(2)}].$$

The argument for $i = 3$ is similar. ☐

4.3.8. We define a second continuous group homomorphism $\xi_{\mathsf{G}}\colon \Xi^2(\mathsf{G}) \to \mathcal{A}(\mathsf{G})$. Consider the continuous map

$$\begin{aligned} (\mathsf{G}(\mathbb{A}_f) \rtimes \mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1) \times_{G^{\mathrm{ad}}(\mathbb{Q}_p)} (G(\mathbb{Q}_p)^2 \rtimes G^{\mathrm{ad}}(\mathbb{Q}_p)) &\to (\mathsf{G}(\mathbb{A}_f) \rtimes \mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1), \\ (((a^p,a_p),b),((x,y),z)) &\mapsto ((a^p, yx^{-1}a_p),b), \end{aligned}$$

which is a group homomorphism because

$$\begin{aligned} yx^{-1}a_p\, {}^b(y'x'^{-1}a'_p) &= yx^{-1}\, {}^{a_p b}(y'x'^{-1})\, a_p\, {}^b a'_p = yx^{-1}\, {}^{xz}(y'x'^{-1})\, a_p\, {}^b a'_p \\ &= y\, {}^z(y'x'^{-1})x^{-1}\, a_p\, {}^b a'_p = (y\, {}^z y')(x\, {}^z x')^{-1}(a_p\, {}^b a'_p) \end{aligned}$$

as $a_p b = xz \in G^{\mathrm{ad}}(\mathbb{Q}_p)$. This sends the closed normal subgroup

$$((Z_{\mathsf{G}}(\mathbb{Q})^- \rtimes \{1\}) \cdot \{(g,g^{-1}) : g \in \mathsf{G}(\mathbb{Q})\}) \times \{((g,g),g^{-1}) : g \in G(\mathbb{Q}_p)\}$$

on the left hand side to the closed normal subgroup

$$(Z_{\mathsf{G}}(\mathbb{Q})^- \rtimes \{1\}) \cdot \{(g,g^{-1}) : g \in \mathsf{G}(\mathbb{Q})\}$$

on the right hand side. Taking quotients on both sides, we obtain a continuous group homomorphism
$$\xi_{\mathsf{G}} \colon \Xi^2(\mathsf{G}) \to \mathcal{A}(\mathsf{G}).$$

Lemma 4.3.9. *The global uniformization map*
$$\Theta \colon \mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_{G,\mu^{-1}}} \mathrm{Gr}_{G,\mu^{-1}} \to \mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}$$
is equivariant with respect to the group homomorphism $\xi_{\mathsf{G}} \colon \Xi^2(\mathsf{G}) \to \mathcal{A}(\mathsf{G})$.

Proof. We need to check equivariance with respect to an arbitrary
$$(((a^p, a_p), b), ((x, y), z)) \in \underline{(\mathsf{G}(\mathbb{A}_f) \rtimes \mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1) \times_{G^{\mathrm{ad}}(\mathbb{Q}_p)} (G(\mathbb{Q}_p)^2 \rtimes G^{\mathrm{ad}}(\mathbb{Q}_p))}(R^\sharp, R^{\sharp+}).$$
Because the subgroup $\{1\} \times \{((g, g), g^{-1}) : g \in G(\mathbb{Q}_p)\}$ acts trivially on both sides, we may reduce to the case when $x = a_p$. Note that this implies $b \mapsto z \in G^{\mathrm{ad}}(\mathbb{Q}_p)$.

We may now write
$$(((a^p, a_p), b), ((x, y), z)) = (((a^p, a_p), 1), ((a_p, y), 1)) \cdot (((1, 1), b), ((1, 1), z)),$$
and check equivariance for both elements separately. For $b \in \underline{\mathsf{G}^{\mathrm{ad}}(\mathbb{Q})^1}(R^\sharp, R^{\sharp+})$, equivariance with respect to $((1, b), (1, b)) \mapsto (1, b)$ follows from functoriality of Θ for the morphism of Shimura data
$$\mathrm{ad}(b) \colon (\mathsf{G}, \mathsf{X}) \to (\mathsf{G}, \mathsf{X}),$$
see [Kim25, Proposition 10.12]. On the other hand, equivariance with respect to $(((a^p, a_p), 1), ((a_p, y), 1)) \mapsto ((a^p, y), 1)$ is the axiom for the Hecke action. $\square$

4.3.10. The map $\xi_{\mathsf{G}} \colon \Xi^2(\mathsf{G}) \to \mathcal{A}(\mathsf{G})$ fits in the commutative diagram
$$\begin{tikzcd}
\mathcal{A}(\mathsf{G}) \arrow[rr, equal] \arrow[d, "{(a,b) \mapsto ((a,b),((1,1),a_p b))}"'] & & \mathcal{A}(\mathsf{G}) \arrow[d, "{(x,y) \mapsto xy}"] \\
\Xi^2(\mathsf{G}) \arrow[r] \arrow[urr, "\xi_{\mathsf{G}}"] & \Sigma^2(G) \arrow[r, "{((x,y),z) \mapsto yz}"] & G^{\mathrm{ad}}(\mathbb{Q}_p)
\end{tikzcd}$$
of locally profinite groups. We also see that the diagram
$$\begin{tikzcd}
\Xi^3(\mathsf{G}) \arrow[r, "{((a,b),((x,y,z),w)) \mapsto ((a^p, y x^{-1} a_p, b),((y,z),w))}"] \arrow[d, "{((a,b),((x,y,z),w)) \mapsto ((a,b),((x,z),w))}"] & \Xi^2(\mathsf{G}) \arrow[d, "\xi_{\mathsf{G}}"] \\
\Xi^2(\mathsf{G}) \arrow[r, "\xi_{\mathsf{G}}"] & \mathcal{A}(\mathsf{G})
\end{tikzcd}$$
commutes.

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Using Lemma 4.3.9 and Lemma 4.3.7, we can push both sides of Θ out along the vertical maps of the commutative diagram
$$\begin{tikzcd}
\Xi^2(\mathsf{G}) \arrow[r, "\xi_{\mathsf{G}}"] \arrow[d] & \mathcal{A}(\mathsf{G}) \arrow[d] \\
\Xi^2(\mathsf{H}) \arrow[r, "\xi_{\mathsf{H}}"] & \mathcal{A}(\mathsf{H})
\end{tikzcd}$$

to obtain

$$\Theta' \colon \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \times_{\mathrm{Bun}_H} \mathrm{Gr}_{H,\mu_H^{-1}} \to \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}.$$

By [Kim25, Proposition 5.12], to prove the theorem, it suffices to check that Θ' is indeed a global uniformization map for (H,Y) in the sense of [Kim25, Definition 5.10]. By construction, the map Θ' is equivariant with respect to the group homomorphism $\xi_{\mathsf{H}} \colon \Xi^2(\mathsf{H}) \to \mathcal{A}(\mathsf{H})$. Because the diagram

$$\begin{tikzcd}
\mathsf{H}(\mathbb{A}_f) \times H(\mathbb{Q}_p) \arrow[r, "\mathrm{pr}_{13} \colon (a,y)\mapsto(a^p,y)"] \arrow[d, "(a,y)\mapsto((a,1),((a_p,y),1))"] & \mathsf{H}(\mathbb{A}_f) \arrow[d, "a\mapsto(a,1)"] \\
\Xi^2(\mathsf{H}) \arrow[r, "\xi_{\mathsf{H}}"] & \mathcal{A}(\mathsf{H})
\end{tikzcd}$$

commutes, we see that Θ' is also equivariant with respect to $\mathrm{pr}_{13} \colon \mathsf{H}(\mathbb{A}_f) \times H(\mathbb{Q}_p) \to \mathsf{H}(\mathbb{A}_f)$.

We now check that the diagrams

$$\begin{tikzcd}
\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \arrow[r, equal] \arrow[d, "(\mathrm{id},\pi_{\mathrm{HT}})"] & \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \arrow[d, "\pi_{\mathrm{HT}}"] \\
\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(2)} \arrow[r, "\mathrm{pr}_2"] \arrow[ur, "\Theta'"'] & \mathrm{Gr}_{H,\mu_H^{-1}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(3)} \arrow[r, "\Theta'\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_{13}"] & \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(2)} \arrow[d, "\Theta'"] \\
\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(2)} \arrow[r, "\Theta'"] & \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}
\end{tikzcd}
\tag{4.3.2}$$

commute. These two diagrams are compatible with the corresponding diagrams for (G,X), which commute by [Kim25, Proposition 5.12]. The commutativity of the diagrams for (H,Y) follows formally once we observe that the natural maps

$$\underline{\mathcal{A}(\mathsf{H})} \times^{\underline{\mathcal{A}(\mathsf{G})}} \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit} \to \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}, \quad \underline{\Xi^i(\mathsf{H})} \times^{\underline{\Xi^i(\mathsf{G})}} \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(i)} \to \mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(i)}$$

are isomorphisms. Indeed, they can be obtained by considering corresponding diagrams for (G,X), and then pushing out along the diagrams of Section 4.3.10 mapping to the corresponding diagrams for H.

Finally, for the absolute Frobenius, we note that the isomorphism $\phi_{\mathrm{Bun}_G} \cong \mathrm{id}_{\mathrm{Bun}_G}$ is functorial in G, and hence the action of $\phi \times \mathrm{id}$ on $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}_{(2)}$ pushes out to the action of $\phi \times \mathrm{id}$ on $\mathbf{Sh}(\mathsf{H},\mathsf{Y})^{?,\diamondsuit}_{(2)}$. Because Θ is assumed to be $\phi \times \mathrm{id}$-invariant, we conclude that Θ' is also $\phi \times \mathrm{id}$-invariant. □

Remark 4.3.11. The two maps $\mathrm{pr}_1, \xi_{\mathsf{G}} \colon \Xi^2(\mathsf{G}) \to \mathcal{A}(\mathsf{G})$ identify the sheaf $\underline{\Xi^2(\mathsf{G})}$ with the fiber product $\underline{\mathcal{A}(\mathsf{G})} \times_{\underline{\mathcal{A}^p(\mathsf{G})}} \underline{\mathcal{A}(\mathsf{G})}$ where we define $\underline{\mathcal{A}^p(\mathsf{G})} = \underline{\mathcal{A}(\mathsf{G})}/\underline{G(\mathbb{Q}_p)}$.[11] Then when we push out the Čech nerve of $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \to \mathrm{Igs}(\mathsf{G},\mathsf{X})$ with the action of the Čech nerve of $\underline{\mathcal{A}(\mathsf{G})} \to \underline{\mathcal{A}^p(\mathsf{G})}$ to $\underline{\mathcal{A}(\mathsf{H})} \to \underline{\mathcal{A}^p(\mathsf{H})}$ we see that the induced map

$$\underline{\mathcal{A}^p(\mathsf{H})} \times^{\underline{\mathcal{A}^p(\mathsf{G})}} \mathrm{Igs}^?(\mathsf{G},\mathsf{X}) \to \mathrm{Igs}^?(\mathsf{H},\mathsf{Y})$$

[11] Note that $G(\mathbb{Q}_p)$ is not typically a closed subgroup of $\mathcal{A}(\mathsf{G})$, so it is better to define the quotient in the world of v-sheaves (or condensed groups).

is an isomorphism. This justifies the heuristic description given after the statement of Theorem 4.3.1.

4.4. **Finishing the construction.** In this section we prove the following theorem.

Theorem 4.4.1. *Fix* $? \in \{\circ, \emptyset\}$*. Let* $(\mathsf{G}_2, \mathsf{X}_2)$ *be a Shimura datum of abelian type and let* v_2 *be a place of the reflex field* E_2 *above a rational prime* p*. Then* $(\mathsf{G}_2, \mathsf{X}_2)$ *admits an Igusa stack for* $?$ *at the place* v_2*.*

4.4.2. Let $(\mathsf{G}_2, \mathsf{X}_2)$ be a Shimura datum of abelian type with reflex field E_2 and fix a p-adic place v_2 of E_2. Recall from the beginning of this section that an *auxiliary Hodge-type Shimura datum* for $(\mathsf{G}_2, \mathsf{X}_2)$ is a pair $((\mathsf{G}, \mathsf{X}), \Xi)$, where (G, X) is a Shimura datum of Hodge type and $\Xi : \mathsf{G}^{\mathrm{der}} \to \mathsf{G}_2^{\mathrm{der}}$ is a central isogeny inducing an isomorphism $(\mathsf{G}^{\mathrm{ad}}, \mathsf{X}^{\mathrm{ad}}) \to (\mathsf{G}_2^{\mathrm{ad}}, \mathsf{X}_2^{\mathrm{ad}})$. The following lemma should be compared to [YZ25, Lemma 4.53].

Lemma 4.4.3. *Let* $((\mathsf{G}, \mathsf{X}), \Xi)$ *be an auxiliary Hodge type Shimura datum with reflex field* E*, and write* $\mathsf{E}' = \mathsf{E} \cdot \mathsf{E}_2$*. Then there exists a Shimura datum* $(\mathsf{G}_3, \mathsf{X}_3)$ *with reflex field* E' *together with maps of Shimura data*

$$(\mathsf{G}_3, \mathsf{X}_3) \to (\mathsf{G}, \mathsf{X}), \quad (\mathsf{G}_3, \mathsf{X}_3) \to (\mathsf{G}_2, \mathsf{X}_2),$$

where the first map induces an isomorphism on derived groups and the second map is an ad-isomorphism.

Proof. This is a direct consequence of [Lov17a, Proposition 3.4.2][12] as we will now explain. Recall that there are natural inclusions $\mathsf{X} \subset \mathsf{X}^{\mathrm{ad}}$ and $\mathsf{X}_2 \subset \mathsf{X}^{\mathrm{ad}}$. By real approximation for G^{ad}, there is an element $g \in \mathsf{G}^{\mathrm{ad}}(\mathbb{Q})$ such that $(\mathrm{Ad}\, g\mathsf{X}) \cap \mathsf{X}_2$ is nonempty. Therefore, after replacing $((\mathsf{G}, \mathsf{X}), \Xi)$ by $((\mathsf{G}, \mathsf{X}), \Xi \circ \mathrm{Ad}\, g)$, we may assume that $\mathsf{X}_2 \cap \mathsf{X}$ is nonempty and thus contains a connected component X^+ of X^{ad}.

We now apply the construction of [Lov17a, Proposition 3.4.2] for the triples

$$(\mathsf{G}^{\mathrm{der}}, \mathsf{X}^+, \mathsf{E}), (\mathsf{G}^{\mathrm{der}}, \mathsf{X}^+, \mathsf{E}'), (\mathsf{G}_2^{\mathrm{der}}, \mathsf{X}^+, \mathsf{E}_2), (\mathsf{G}_2^{\mathrm{der}}, \mathsf{X}^+, \mathsf{E}'),$$

which gives us Shimura data

$$(\mathsf{B}, \mathsf{X}), (\mathsf{B}', \mathsf{X}'), (\mathsf{B}_2, \mathsf{X}_2), (\mathsf{B}_2', \mathsf{X}_2'),$$

respectively, with reflex fields $\mathsf{E}, \mathsf{E}', \mathsf{E}_2, \mathsf{E}'$, respectively. By construction, these come equipped with morphisms of Shimura data

$$(\mathsf{B}, \mathsf{X}) \to (\mathsf{G}, \mathsf{X}), \quad (\mathsf{B}_2, \mathsf{X}_2) \to (\mathsf{G}_2, \mathsf{X}_2)$$

that induce isomorphisms of derived groups. By [Lov17a, Proposition 3.4.2.(2)], there are morphisms of Shimura data

$$(\mathsf{B}', \mathsf{X}') \to (\mathsf{B}, \mathsf{X}), \quad (\mathsf{B}_2', \mathsf{X}_2') \to (\mathsf{B}_2, \mathsf{X}_2)$$

that are isomorphisms on derived groups. By [Lov17a, Proposition 3.4.2.(1)] there is a morphism of Shimura data (induced by Ξ)

$$(\mathsf{B}', \mathsf{X}') \to (\mathsf{B}_2', \mathsf{X}_2')$$

[12] We are citing an unpublished preprint here, however the proposition we need is simply a convenient summary of the results in the (published) [Lov17b, Section 4.6].

which induces a central isogeny on derived groups, it is thus an ad-isomorphism. The lemma is proved by taking $(\mathsf{G}_3, \mathsf{X}_3) = (\mathsf{B}', \mathsf{X}')$ with its corresponding maps to (G, X) and $(\mathsf{G}_2, \mathsf{X}_2)$. For the convenience of the reader, we record the following diagram summarizing the argument:

$$\begin{tikzcd}
(\mathsf{B}', \mathsf{X}') \arrow[r] \arrow[d] & (\mathsf{B}'_2, \mathsf{X}'_2) \arrow[r] & (\mathsf{B}_2, \mathsf{X}_2) \arrow[r] & (\mathsf{G}_2, \mathsf{X}_2) \\
(\mathsf{B}, \mathsf{X}) \arrow[rrr] & & & (\mathsf{G}, \mathsf{X}).
\end{tikzcd}$$

□

Proof of Theorem 4.4.1. By the proof of [KP18, Lemma 4.6.22], we can find an auxiliary Hodge type Shimura datum $((\mathsf{G}, \mathsf{X}), \Xi)$ with reflex field E such that every prime above p in E_2 splits completely in the extension $\mathsf{E}' = \mathsf{E} \cdot \mathsf{E}_2$. Choose $(\mathsf{G}_2, \mathsf{X}_2) \leftarrow (\mathsf{G}_3, \mathsf{X}_3) \to (\mathsf{G}, \mathsf{X})$ as in Lemma 4.4.3. Choose a place w' of E' lifting v_2, which induces a place w of E, and note that we have an equality of local reflex fields $\mathsf{E}'_{w'} = \mathsf{E}_{2,v_2}$.

The existence of the Igusa stack for (G, X) is [DvHKZ24a, Theorem I] and [Kim25, Theorem D] for $? = \circ$, $? = \emptyset$, respectively. The existence of the Igusa stack for $(\mathsf{G}_3, \mathsf{X}_3)$ now follows from Theorem 4.1.5 and the existence of the Igusa stack for $(\mathsf{G}_2, \mathsf{X}_2)$ is then a consequence of Theorem 4.3.1. □

4.5. **A finite level Igusa stack.** Let (G, X) be a Shimura datum of abelian type with reflex field E and let v be a place of E above a prime number p. Let $G = \mathsf{G} \otimes \mathbb{Q}_p$, $E = \mathsf{E}_v$ and μ be as before. Then by Theorem 4.4.1, (G, X) admits an Igusa stack $\mathrm{Igs}^{?}(\mathsf{G}, \mathsf{X}) \to \mathrm{Bun}_G$ for ? in $\{\circ, \emptyset\}$ at the place v. The goal of this section is to construct a finite-level Igusa stack $\mathrm{Igs}^{?}_{K^p}(\mathsf{G}, \mathsf{X})$ which fits in a Cartesian diagram with $\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}$. This is more difficult in the case of abelian type Shimura data than in the case of Hodge type Shimura data, because K^p does not generally act freely on $\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}$ and so we cannot simply take the stack quotient.

Theorem 4.5.1. *Let $K^p \subseteq \mathsf{G}(\mathbb{A}_f^p)$ be a neat compact open subgroup. For $? \in \{\circ, \emptyset\}$, there is a v-stack $\mathrm{Igs}^{?}_{K^p}(\mathsf{G}, \mathsf{X})$ fitting into the 2-commutative diagram*

$$\begin{tikzcd}
\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \arrow[r] \arrow[d] & \mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \arrow[r] \arrow[d] & \mathrm{Gr}_{G,\mu^{-1}} \arrow[d] \\
\mathrm{Igs}^{?}(\mathsf{G}, \mathsf{X}) \arrow[r] & \mathrm{Igs}^{?}_{K^p}(\mathsf{G}, \mathsf{X}) \arrow[r] & \mathrm{Bun}_G,
\end{tikzcd}$$

where both squares are 2-Cartesian. Moreover, the absolute Frobenius acts trivially on $\mathrm{Igs}^{?}_{K^p}(\mathsf{G}, \mathsf{X})$ with the trivialization being compatible with that of Bun_G.

4.5.2. To construct the Igusa stack at finite level, we will use the following lemma. Recall from Theorem 2.1.7 that giving an Igusa stack $\mathrm{Igs}^{?}(\mathsf{G}, \mathsf{X})$ is equivalent to giving a uniformization map

$$\Theta \colon \mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \to \mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}$$

in the sense of Definition 2.1.6.

Lemma 4.5.3. *For $? \in \{\circ, \emptyset\}$, there is a unique map Θ_{K^p} filling in the diagram*

$$\begin{tikzcd} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \arrow[r, "\Theta"] \arrow[d] & \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \arrow[d] \\ \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \arrow[r, dashed, "\Theta_{K^p}"] & \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}. \end{tikzcd}$$

Moreover, the map Θ_{K^p} intertwines the action of $\phi \times \mathrm{id}$ on the source with the action of id. Furthermore, the following diagrams commute.

$$\begin{tikzcd} \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \arrow[rr, equal] \arrow[d, "(\mathrm{id},\pi_{\mathrm{HT},K^p})"'] & & \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \arrow[d, "\pi_{\mathrm{HT},K^p}"] \\ \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \arrow[urr, "\Theta_{K^p}"] \arrow[rr, "\mathrm{pr}_2"] & & \mathrm{Gr}_{G,\mu^{-1}} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \arrow[r, "\Theta_{K^p}\times\mathrm{id}"] \arrow[d, "\mathrm{pr}_{1,3}"] & \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \arrow[d, "\Theta_{K^p}"] \\ \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \arrow[r, "\Theta_{K^p}"] & \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}. \end{tikzcd}$$

Proof. The group $\underline{\mathsf{G}(\mathbb{A}_f)} \times \underline{G(\mathbb{Q}_p)}$ acts on $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}}$ and the map Θ is equivariant for the action of $\underline{\mathsf{G}(\mathbb{A}_f)} \times \underline{G(\mathbb{Q}_p)}$ along $\mathrm{pr}_{1,3}\colon \underline{\mathsf{G}(\mathbb{A}_f)} \times \underline{G(\mathbb{Q}_p)} = \underline{\mathsf{G}(\mathbb{A}_f^p)} \times \underline{G(\mathbb{Q}_p)} \times \underline{G(\mathbb{Q}_p)} \to \underline{\mathsf{G}(\mathbb{A}_f^p)} \times \underline{G(\mathbb{Q}_p)}$. By [Kim25, Lemma 6.2], the closed subgroup

$$\underline{(\mathrm{id}\times\Delta)(Z_{\mathsf{G}}(\mathbb{Q})^-)} \subseteq \underline{\mathsf{G}(\mathbb{A}_f^p)} \times \underline{G(\mathbb{Q}_p)} \times \underline{G(\mathbb{Q}_p)}$$

acts trivially on $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}}$. Since $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \to \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}$ is a torsor for the profinite group (see [KSZ21, Section 1.5.8])

$$\frac{K^p \cdot Z_{\mathsf{G}}(\mathbb{Q})^-}{Z_{\mathsf{G}}(\mathbb{Q})^-} \subseteq \frac{\mathsf{G}(\mathbb{A}_f)}{Z_{\mathsf{G}}(\mathbb{Q})^-},$$

it follows that

$$\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}} \to \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,\mu^{-1}}$$

is a torsor for the profinite group

$$\frac{(K^p \times \{1\} \times \{1\}) \cdot (\mathrm{id}\times\Delta)(Z_{\mathsf{G}}(\mathbb{Q})^-)}{(\mathrm{id}\times\Delta)(Z_{\mathsf{G}}(\mathbb{Q})^-)} \subseteq \frac{\mathsf{G}(\mathbb{A}_f^p) \times G(\mathbb{Q}_p) \times G(\mathbb{Q}_p)}{(\mathrm{id}\times\Delta)(Z_{\mathsf{G}}(\mathbb{Q})^-)}.$$

This maps isomorphically to $(K^p \cdot Z_{\mathsf{G}}(\mathbb{Q})^-)/Z_{\mathsf{G}}(\mathbb{Q})^-$ under $\mathrm{pr}_{1,3}$, showing that Θ descends to a map Θ_{K^p}. The statement about the ϕ-action follows formally from the statement about the ϕ-action for Θ, which holds by [Kim25, Proposition 5.12].

The two diagrams that we have to show are commutative, sit in commutative diagrams with the same diagrams without the subscripts K^p as in [Kim25, Definition 5.10.(3)]; these latter diagrams moreover commute by [Kim25, Proposition 5.12]. Since the maps from the objects without subscript K^p to the objects with subscript K^p are v-covers, it follows that the diagrams at level K^p must also commute. $\square$

As a corollary, we have the following result (which, together with Theorem 4.4.1 and Proposition 5.1.9, proves Theorem I from the introduction).

Corollary 4.5.4. *For $? \in \{\circ, \emptyset\}$, there is a v-stack $\mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X})$ fitting into the following 2-commutative diagram*

$$\begin{array}{ccccc} \mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} & \longrightarrow & \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} & \xrightarrow{\pi^?_{\mathrm{HT}}} & \mathrm{Gr}_{G,\mu^{-1}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Igs}^?(\mathsf{G},\mathsf{X}) & \longrightarrow & \mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X}) & \xrightarrow{\overline{\pi}^?_{\mathrm{HT}}} & \mathrm{Bun}_G, \end{array}$$

where both squares are 2-Cartesian. Moreover, the absolute Frobenius acts trivially on $\mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X})$ where there is a canonical trivialization that is compatible with that of Bun_G.

Proof. This follows from Lemma 4.5.3 together with arguments in [Kim25, Proposition 5.12]. □

5. Cohomology of Shimura varieties

In this section we generalize many of the results of [DvHKZ24a, Section 8, Section 9] to abelian type Shimura varieties. We compute the dualizing sheaf of the Igusa stack in Section 5.1, define the Igusa sheaf and study its (conjectural) properties in Section 5.2. In Section 5.3, we compute the (compactly supported) cohomology of the Shimura variety in terms of a Hecke operator acting on the Igusa sheaf. In Section 5.4 we prove a version of the Mantovan product formula, in Section 5.5 we prove a compatibility statement of the cohomology of Shimura varieties with the Fargues–Scholze local Langlands correspondence, in Section 5.6 we prove the Eichler–Shimura relation and in Section 5.7 we prove the split local plectic conjecture.

5.1. **The dualizing sheaf of the Igusa stack.** Let (G,X) be a Shimura datum of abelian type with reflex field E and let v be a place of E above a prime number p. Let $G = \mathsf{G}\otimes\mathbb{Q}_p$, $E = \mathsf{E}_v$ and μ be as before and let $K^p \subset \mathsf{G}(\mathbb{A}^p_f)$ be a neat compact open subgroup. For $?$ in $\{\circ, \emptyset\}$, we study the Igusa stack $\mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X})$ of Corollary 4.5.4. The goal of this section is to prove that $\mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X})$ is an ℓ-cohomologically smooth v-stack of ℓ-cohomological dimension zero with trivial dualizing sheaf, generalizing [DvHKZ24a, Theorem 8.3.1]. This is more difficult in the case of abelian type Shimura data than in the case of Hodge type Shimura data, because K^p does not generally act freely on $\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}$.

Lemma 5.1.1. *For $? \in \{\emptyset, \circ\}$, the morphism $\overline{\pi}^?_{\mathrm{HT}}\colon \mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X}) \to \mathrm{Bun}_{G,\mu^{-1}}$ is compactifiable, representable in locally spatial diamonds, and has $\dim.\mathrm{trg}\,\overline{\pi}_{\mathrm{HT}} < \infty$. If $? = \emptyset$, then it is moreover partially proper.*

Proof. The morphism $\pi^?_{\mathrm{HT}} : \mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \to \mathrm{Gr}_{G,\mu^{-1}}$ is separated, compactifiable, representable in locally spatial diamonds, and has $\dim.\mathrm{trg}\,\pi_{\mathrm{HT}} < \infty$ since it is a map between pro-(finite-étale) covers of rigid analytic varieties. We also note that $\mathrm{Gr}_{G,\mu^{-1}} \to \mathrm{Bun}_{G,\mu^{-1}}$ is surjective in the pro-étale topology, see [DvHKZ24a, Corollary 6.4.2]. Using [Sch22, Proposition 10.11.(ii), Proposition 13.4.(iv)] we see that $\overline{\pi}^{?}_{\mathrm{HT}}$ is separated and representable in locally spatial diamonds. To check the finiteness of $\dim.\mathrm{trg}\,\overline{\pi}^{?}_{\mathrm{HT}}$ it is enough to check on geometric points on $\mathrm{Bun}_{G,\mu^{-1}}$ in the image of $\mathrm{Igs}^{?}_{K^p}(\mathsf{G},\mathsf{X})$. By surjectivity of $\mathrm{Gr}_{G,\mu^{-1}} \to \mathrm{Bun}_{G,\mu^{-1}}$ in the pro-étale topology, such a point lifts to a point of $\mathrm{Gr}_{G,\mu^{-1}}$, and therefore it follows from $\dim.\mathrm{trg}\,\pi^{?}_{\mathrm{HT}} < \infty$.

Now take $? = \emptyset$. The canonical compactification of $\overline{\pi}_{\mathrm{HT}}$ base changes to the canonical compactification of π_{HT} by [Sch22, Proposition 18.6], hence partial properness of π_{HT} implies partial properness of $\overline{\pi}_{\mathrm{HT}}$. □

5.1.2. Next, we prove that the dualizing sheaf of the Igusa stack is constant. To perform this computation, we work with the nuclear sheaf theory developed in [Man26]. In what follows, we let Λ be a $\mathbb{Z}_\ell$-algebra in which ℓ is nilpotent. In this case, [Man26, Proposition 3.20] states that the nuclear sheaf theory of [Man26] agrees with the étale sheaf theory of [Sch22], where the six functors also agree by construction. Note that the dualizing sheaf indeed makes sense because both $\overline{\pi}^{?}_{\mathrm{HT}}$ and $\mathrm{Bun}_{G,\mu^{-1}} \to \mathrm{Spd}\,\mathbb{F}_p$ admit !-functors (they are ℓ-fine in the sense of [Man26, Definition 5.8]) see Lemma 5.1.8.

Definition 5.1.3. Let G be a locally profinite group that is unimodular, and let X be a v-stack. Write $\Gamma_G = \mathrm{im}(\delta_G\colon \mathrm{Out}(G) \to \mathbb{Q}^{\times}_{>0})$, see Section 2.2.3. We say that a $\underline{G}$-gerbe $Y \to X$ for the v-topology is *unimodular* when the corresponding $\mathrm{Out}(\underline{G})$-torsor (see for example [Gir71, Section IV.2.2] or [Bre94, Section 2.10])

$$\mathrm{Isom}(Y, X \times [*/\underline{G}]) \to X$$

has the property that its pushout along $\delta_G\colon \mathrm{Out}(\underline{G}) \twoheadrightarrow \underline{\Gamma_G}$ is a trivial $\underline{\Gamma_G}$-torsor over X.

Example 5.1.4. Here are examples of unimodular and non-unimodular gerbes.

(1) If G is compact, then $\Gamma_G = 1$ and hence all $\underline{G}$-gerbes are unimodular.
(2) If k is a discrete perfect field of characteristic p, and a $\underline{G}$-gerbe $Y \to \mathrm{Spd}\,k$ splits over $\mathrm{Spd}\,\bar{k}$, it is unimodular. This is because the $\underline{\Gamma_G}$-torsor over k that splits over $\bar{k}$ corresponds to an element of
$$H^1(\bar{k}/k, \Gamma_G) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(\bar{k}/k), \Gamma_G) = 0,$$
where the last equality holds because $\Gamma_G \subseteq \mathbb{Q}^{\times}_{>0}$ is torsion free and has the discrete topology.
(3) The $\underline{\mathbb{Q}_p}$-gerbe over $[\mathrm{Spd}\,\overline{\mathbb{F}}_p/\phi^{\mathbb{Z}}]$ constructed by descending $\mathrm{Spd}\,\overline{\mathbb{F}}_p \times [*/\underline{\mathbb{Q}_p}]$ along the multiplication map $p\colon [*/\underline{\mathbb{Q}_p}] \to [*/\underline{\mathbb{Q}_p}]$ is not unimodular, see Example 2.2.5.
(4) Let $1 \to K \to G \to H \to 1$ be a short exact sequence of locally profinite groups, where the conjugation action of G on $\mathrm{Haar}(K, \mathbb{Q})$ is trivial (i.e., $\delta_G = \delta_H$ as characters of G). Then the $\underline{K}$-gerbe $[*/\underline{G}] \to [*/\underline{H}]$ is unimodular.

This is because the $\mathrm{Out}(\underline{K})$-torsor is obtained by pushing out the $\underline{H}$-torsor $* \to [*/\underline{H}]$ along $\underline{H} \to \mathrm{Out}(\underline{K})$, where the composition

$$H \to \mathrm{Out}(K) \xrightarrow{\delta_K} \mathbb{Q}_{>0}^{\times}$$

is trivial.

Lemma 5.1.5. *Let G be a locally profinite group that is locally pro-prime-to-ℓ and unimodular. Let $\pi\colon Y \to X$ be a unimodular $\underline{G}$-gerbe for the v-topology, and assume that π is ℓ-fine. Then π is ℓ-cohomologically smooth, and moreover there exists a (noncanonical) isomorphism $\pi^!\Lambda \cong \Lambda[0]$.*

Remark 5.1.6. In general, even if Y is not unimodular, $\pi^!\Lambda$ is canonically identified with the constant sheaf $\mathrm{Haar}(G,\Lambda)^*$ twisted by the $\underline{\Gamma_G}$-torsor corresponding to the gerbe.

Proof of Lemma 5.1.5. We choose a trivialization of the $\underline{\Gamma_G}$-torsor

$$P = \mathrm{Isom}(Y, X \times [*/\underline{G}]) \times^{\mathrm{Out}(\underline{G})} \underline{\Gamma_G}.$$

Let $\tilde{X} \to X$ be a v-cover by a perfectoid space $\tilde{X}$ that splits π, so that there is an isomorphism

$$f\colon \tilde{X} \times_X Y \cong \tilde{X} \times [*/\underline{G}]$$

over $\tilde{X}$. Here, we note that this defines a trivialization of $P_{\tilde{X}}$, which may differ with our chosen trivialization of P by an element of $\mathrm{Fun}_{\mathrm{cts}}(|\tilde{X}|, \Gamma_G)$. Upon choosing a set-theoretic section of $\mathrm{Aut}(G) \twoheadrightarrow \Gamma_G$, we may modify f so that the two trivializations of P and $P_{\tilde{X}}$ are compatible. By [Man26, Theorem 10.13, Example 10.11.(b)], the map $[*/\underline{G}] \to *$ is ℓ-cohomologically smooth, and hence so is $\tilde{X} \times_X Y \to \tilde{X}$. Because π is assumed to be ℓ-fine, it is also ℓ-cohomologically smooth by [Man26, Lemma 8.7.(ii)].

Once π is ℓ-cohomologically smooth, we may compute the dualizing sheaf $\pi^!\Lambda$ v-locally using [Man26, Proposition 8.5.(ii).(a)]. As discussed in [HKW22, Example 4.2.4], the dualizing sheaf of $[*/\underline{G}] \to *$ is canonically identified with $\mathrm{Haar}(G,\Lambda)^*$. Here, we note that $\mathrm{Haar}(G,\Lambda)^* \in \mathcal{D}([*/\underline{G}],\Lambda)$ is a constant sheaf because G is unimodular. Write $\tilde{X}^{(2)} = \tilde{X} \times_X \tilde{X}$ and

$$\alpha\colon \tilde{X}^{(2)} \times [*/\underline{G}] \cong \tilde{X} \times_X (\tilde{X} \times_X Y) \xrightarrow{\mathrm{swap}_{23}} (\tilde{X} \times_X Y) \times_X \tilde{X} \cong \tilde{X}^{(2)} \times [*/\underline{G}]$$

for the descent datum so that

$$\begin{tikzcd} \tilde{X}^{(2)} \times [*/\underline{G}] \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{pr}_1"'] & & \tilde{X}^{(2)} \times [*/\underline{G}] \arrow[dl, "\mathrm{pr}_1"] \\ & \tilde{X}^{(2)} & \end{tikzcd}$$

naturally commutes. It remains to verify that the induced map

$$\mathrm{Haar}(G,\Lambda)^* \cong \mathrm{pr}_1^!\Lambda = \alpha^!\mathrm{pr}_1^!\Lambda \cong \alpha^!\,\mathrm{Haar}(G,\Lambda)^* \cong \mathrm{Haar}(G,\Lambda)^*$$

is the identity map in $\mathcal{D}(\tilde{X} \times [*/\underline{G}],\Lambda)$.

Because $\mathrm{Haar}(G,\Lambda)^*$ is a free Λ-module of rank 1, such an automorphism corresponds to an element of $\mathrm{Fun}_{\mathrm{cts}}(|\tilde{X}^{(2)} \times [*/\underline{G}]|, \Lambda^\times) = \mathrm{Fun}_{\mathrm{cts}}(|\tilde{X}^{(2)}|, \Lambda^\times)$. Therefore we may compute this element after pulling back along each geometric point

$x = \mathrm{Spa}(C, C^+) \to \tilde{X}^{(2)}$. Write $q\colon x \to x \times [*/\underline{G}]$ for the natural map. Because every $\underline{G}$-torsor over x is trivial, there exists a 2-morphism filling in

$$\begin{tikzcd} & x \arrow[dl, "q"'] \arrow[dr, "q"] & \\ x \times [*/\underline{G}] \arrow[rr, "\alpha_x"'] & & x \times [*/\underline{G}]. \end{tikzcd}$$

By considering the automorphism groups, we obtain an automorphism $\beta_x\colon G \to G$ of topological groups together with an isomorphism $\alpha_x \cong [*/\beta_x]$. Then the automorphism

$$\mathrm{Haar}(G, \Lambda)^* \cong \mathrm{pr}_1^! \Lambda = \beta_x^! \mathrm{pr}_1^! \Lambda \cong \beta_x^! \,\mathrm{Haar}(G, \Lambda)^* \cong \mathrm{Haar}(G, \Lambda)^*$$

is the one induced by β_x, and hence is multiplication by $\delta_G(\beta_x)$. Because we have chosen the isomorphism $f\colon \tilde{X} \times_X Y \cong \tilde{X} \times [*/\underline{G}]$ to respect the trivialization of P, the automorphism β_x is contained in the kernel of

$$\mathrm{Aut}(G) \twoheadrightarrow \mathrm{Out}(G) \xrightarrow{\delta_G} \Gamma_G,$$

and therefore $\delta_G(\beta_x) = 1$. □

Lemma 5.1.7. *The map $\pi\colon [\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Spd}\, E$ is ℓ-cohomologically smooth, and moreover the dualizing sheaf is isomorphic to $\Lambda(d)[2d]$, where $d = \langle 2\rho, \mu \rangle = \dim \mathsf{X}$.*

Proof. We first verify that $\pi\colon [\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Spd}\, E$ is ℓ-fine. By [Man26, Lemma 10.8], the map $* \to [*/\underline{G(\mathbb{Q}_p)}]$ is fdcs and admits universal ℓ-codescent.[13] Hence its base change $\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \to [\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{G(\mathbb{Q}_p)}]$ is also fdcs and admits universal ℓ-codescent. On the other hand, the map $\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \to \mathrm{Spd}\, E$ is a limit of smooth rigid analytic varieties along finite étale maps, hence is fdcs. It now follows from [Man26, Definition 5.8] that π is ℓ-fine.

We now compute the dualizing sheaf. Choose a sufficiently small open compact subgroup $K_p \subseteq G(\mathbb{Q}_p)$ and consider the locally profinite groups

$$A = G(\mathbb{Q}_p) \times \frac{K^p Z_{\mathsf{G}}(\mathbb{Q})^-}{Z_{\mathsf{G}}(\mathbb{Q})^-} \twoheadrightarrow B = \frac{(K^p G(\mathbb{Q}_p)) Z_{\mathsf{G}}(\mathbb{Q})^-}{Z_{\mathsf{G}}(\mathbb{Q})^-} \supseteq \frac{(K^p K_p) Z_{\mathsf{G}}(\mathbb{Q})^-}{Z_{\mathsf{G}}(\mathbb{Q})^-},$$

where the first map is multiplication (noting that the image of the two factors commute with each other), and the second map is an inclusion of an open subgroup. These groups naturally act on $\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}$, and taking the respective quotients recovers

$$[\mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{G(\mathbb{Q}_p)}] \xrightarrow{f} [\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{B}] \leftarrow \mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}.$$

Note that $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{?,\diamondsuit} \to [\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{B}]$ is an étale quotient because it corresponds to an open subgroup, hence $\rho\colon [\mathbf{Sh}(\mathsf{G}, \mathsf{X})^{?,\diamondsuit}/\underline{B}] \to \mathrm{Spd}\, E$ is ℓ-fine. Applying [Man26, Lemma 5.10.(iv)], we see that f is ℓ-fine as well.

[13] While this is stated only for profinite groups, we can simply choose a pro-p compact open subgroup $K_p \subseteq \underline{G(\mathbb{Q}_p)}$ and observe that $* \to [*/\underline{K_p}]$ and $[*/\underline{K_p}] \to [*/\underline{G(\mathbb{Q}_p)}]$ are both fdcs and admit universal ℓ-codescent, because the second map is étale. Then [LZ24, Lemma 3.3.2.(4)] shows that their composition also admits universal ℓ-codescent.

We note that f is the base change of

$$\bar{f}\colon [*/\underline{A}] \to [*/\underline{B}].$$

The group A is unimodular because it is the product of two unimodular groups. The group B is unimodular because it is an open subgroup of $\mathsf{G}(\mathbb{A}_f)/Z_{\mathsf{G}}(\mathbb{Q})^-$, which is unimodular as it is a central quotient of the unimodular group $\mathsf{G}(\mathbb{A}_f)$. It follows from Example 5.1.4.(4) that $\bar{f}$ is a unimodular gerbe for $\ker(A \to B)$, and hence its base change f is also a unimodular gerbe. The kernel of $A \to B$ is a closed subgroup of $G(\mathbb{Q}_p)$, hence locally pro-p, and therefore by Lemma 5.1.5 we obtain an isomorphism

$$\pi^!\Lambda = f^!\rho^!\Lambda \cong f^*\rho^!\Lambda \otimes_\Lambda f^!\Lambda \cong f^*\rho^!\Lambda.$$

On the other hand, by [DvHKZ24a, Remark 8.3.5] the dualizing sheaf $\rho^!\Lambda$ is isomorphic to $\Lambda(d)[2d]$ because $[\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}/\underline{B}]$ is an étale quotient of a smooth rigid analytic variety.

Finally, to show that π is ℓ-cohomologically smooth, we note that $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{?,\diamondsuit}$ is ℓ-cohomologically smooth over $\operatorname{Spd} E$. Because $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{?,\diamondsuit} \to [\mathbf{Sh}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}/\underline{B}]$ is étale and f is ℓ-cohomologically smooth by Lemma 5.1.5, we may use [Man26, Lemma 8.7.(ii)] to conclude that $[\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}/\underline{G(\mathbb{Q}_p)}]$ is ℓ-cohomologically smooth over $\operatorname{Spd} E$. □

Lemma 5.1.8. *The map* $\pi\colon \mathrm{Bun}_G \to \operatorname{Spd}\mathbb{F}_p$ *is* ℓ*-cohomologically smooth with dualizing sheaf* $\pi^!\Lambda \cong \Lambda[0]$.

Proof. The ℓ-fineness appears in [FS24, Theorem IV.1.19]; while the theorem is stated over $\overline{\mathbb{F}}_p$, the proof also works over $\mathbb{F}_p$. For ℓ-cohomological smoothness, we use [Man26, Lemma 8.7.(ii)] to reduce to the statement over $\overline{\mathbb{F}}_p$. The dualizing complex over $\overline{\mathbb{F}}_p$ was computed in [HI25, Proposition 3.18], and it remains to descend it to $\mathbb{F}_p$. Because the union of $\mathrm{Bun}_G^{[b]} \subseteq \mathrm{Bun}_G$ over all basic $b \in B(G)$ intersects all connected components of Bun_G, it suffices to show that the dualizing complex of $\mathrm{Bun}_G^{[b]} \to \operatorname{Spd}\mathbb{F}_p$ is trivial. Because this map is a $\underline{G_b(\mathbb{Q}_p)}$-gerbe, it now follows from Example 5.1.4.(2) and Lemma 5.1.5. □

Proposition 5.1.9. *The map* $\mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X}) \to \operatorname{Spd}\mathbb{F}_p$ *is* ℓ*-cohomologically smooth with dualizing sheaf isomorphic to* $\Lambda[0]$.

Proof. We first compute the dualizing sheaf. The second half of the proof of [DvHKZ24a, Theorem 8.3.1] works almost verbatim. Consider the Cartesian diagram

$$\begin{tikzcd}
{[\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^{?,\diamondsuit}/(\underline{G(\mathbb{Q}_p)} \times \phi^{\mathbb{Z}})]} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"] & {[\mathrm{Gr}_{G,\mu^{-1}}/(\underline{G(\mathbb{Q}_p)} \times \phi^{\mathbb{Z}})]} \arrow[d, "g"] \\
\mathrm{Igs}^?_{K^p}(\mathsf{G},\mathsf{X}) \arrow[r, "f"] & \mathrm{Bun}_{G,\mu^{-1}}.
\end{tikzcd}$$

By Lemma 5.1.8 it suffices to show that $f^!\Lambda \cong \Lambda[0]$. The map g is ℓ-cohomologically smooth by combining [FS24, Theorem IV.1.19] with [Man26, Lemma 8.7.(i)]. Hence by [Man26, Proposition 8.5.(ii)] we have

$$\tilde{g}^* f^! \Lambda \cong \tilde{f}^! g^* \Lambda = \tilde{f}^! \Lambda \cong \Lambda[0]$$

where the last isomorphism follows from [DvHKZ24a, Lemma 8.3.4], Lemma 5.1.7. This shows that $f^!\Lambda$ is an invertible sheaf in degree zero, and by adjunction we produce a map $f^!\Lambda = \tau_{\leq 0} f^! \Lambda \to \tau_{\leq 0} R\tilde{g}_* \Lambda \cong \Lambda[0]$ that we check is an isomorphism upon pulling back along $\tilde{g}$. Here, the isomorphism $\tau_{\leq 0} R\tilde{g}_* \Lambda \cong \Lambda[0]$ comes from the fact that the fiber of $\tilde{g}$ over a (C, C^+)-point, which is a fiber of g, is a relative admissible locus over $\mathcal{X}_{\mathrm{FF}}(C, C^+)_E$ and hence connected by [GL22, Theorem 1.1].

To show that $\pi\colon \mathrm{Igs}^?_{K^p}(\mathsf{G}, \mathsf{X}) \to \mathrm{Spd}\, \mathbb{F}_p$ is ℓ-cohomologically smooth, we simply note that $[\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})^{?,\diamond} / \underline{G(\mathbb{Q}_p)}]$ is ℓ-cohomologically smooth over $\mathrm{Spd}\, \mathbb{F}_p$ by Lemma 5.1.7 and [Sch22, Proposition 24.5], and also ℓ-cohomologically smooth over $\mathrm{Igs}^?_{K^p}(\mathsf{G}, \mathsf{X})$. Because π is already ℓ-fine by Lemma 5.1.1, we may use [Man26, Lemma 8.7.(ii)] to conclude that π is ℓ-cohomologically smooth. $\square$

5.2. **Conjectures about the Igusa sheaf.** In this section, we define the Igusa sheaf and state some conjectures.

5.2.1. Let Λ be a $\mathbb{Z}_\ell$ algebra in which ℓ is nilpotent. Let W be a continuous representation of K^p on a finite free Λ-module that factors through the image of K^p under $\mathsf{G}(\mathbb{A}_f) \to \mathsf{G}^c(\mathbb{A}_f)$. As in [KSZ21, Section 1.5.8], we can use W to define a local system $\mathbb{W}$ on $\mathbf{Sh}_{K_p K^p}(\mathsf{G}, \mathsf{X})$ for each compact open subgroup K_p of $G(\mathbb{Q}_p)$; these are compatible with one another as the level changes. We will refer to local systems defined in this way from such a W as *standard Λ-local systems.*

5.2.2. By Corollary 4.5.4, the covering group of $\mathrm{Igs}(\mathsf{G}, \mathsf{X}) \to \mathrm{Igs}_{K^p}(\mathsf{G}, \mathsf{X})$ is the same as the covering group of $\mathbf{Sh}(\mathsf{G}, \mathsf{X})^\diamond \to \mathbf{Sh}_{K^p}(\mathsf{G}, \mathsf{X})^\diamond$. Thus we can descend a standard Λ-local system $\mathbb{W}$ to Igs_{K^p}, and we will refer to both the local system and its descent as $\mathbb{W}$. For such a $\mathbb{W}$ we define (using Lemma 5.1.1)

$$\mathcal{F}_{c,\mathbb{W}} := \overline{\pi}_{\mathrm{HT},!} \mathbb{W} \quad \mathcal{F}_{\mathbb{W}} := R\overline{\pi}_{\mathrm{HT},*} \mathbb{W}$$
$$\mathcal{F}^\circ_{c,\mathbb{W}} := \overline{\pi}^\circ_{\mathrm{HT},!} \mathbb{W} \quad \mathcal{F}^\circ_{\mathbb{W}} := R\overline{\pi}^\circ_{\mathrm{HT},*} \mathbb{W}.$$

Remark 5.2.3. The sheaf denoted by $\mathcal{F}$ in [DvHKZ24a, Section 8.4.9] is the sheaf $\mathcal{F}^\circ_\Lambda$ defined above. Note that if $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$ is proper, then $\mathrm{Igs}^\circ(\mathsf{G}, \mathsf{X}) = \mathrm{Igs}(\mathsf{G}, \mathsf{X})$ and moreover π_{HT}, hence $\overline{\pi}_{\mathrm{HT}}$, is proper. This shows that $\mathcal{F}^\circ_{\mathbb{W}} = \mathcal{F}_{\mathbb{W}} = \mathcal{F}_{c,\mathbb{W}} = \mathcal{F}^\circ_{c,\mathbb{W}}$.

We have the following conjecture. Let k be an algebraic closure of $\mathbb{F}_p$.

Conjecture 5.2.4. *Let $\mathbb{W}$ be a standard Λ-local system.*

- (*i*) *For $? \in \{\emptyset, \circ\}$, the sheaves $\mathcal{F}^?_{c,\mathbb{W}}, \mathcal{F}^?_{\mathbb{W}}$ are universally locally acyclic with respect to* $\mathrm{Bun}_{G,\mu^{-1},k} \to \mathrm{Spd}\, k$.
- (*ii*) *The sheaf $\mathcal{F}_{c,\mathbb{W}}$ lies in ${}^pD^{\leq 0}(\mathrm{Bun}_{G,\mu^{-1},k}, \Lambda)$ for the perverse t-structure on $\mathcal{D}(\mathrm{Bun}_{G,\mu^{-1},k}, \Lambda)$ of* [DvHKZ24a, Proposition 8.1.5].

Remark 5.2.5. Assume that $\mathbb{W} = \Lambda$. For many PEL type Shimura varieties, both parts of Conjecture 5.2.4 follow from [HL23, Proposition 3.7] using Lemma 5.2.8. If (G, X) is a Shimura variety of Hodge type, then Conjecture 5.2.4(i) for $? = \circ$

is [DvHKZ24a, Corollary 8.5.4]. If $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ is moreover compact, then Conjecture 5.2.4(ii) is proved under mild assumptions in [DvHKZ24a, Theorem 8.6.3].

Proposition 5.2.6. *If $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ is proper, then Conjecture 5.2.4(i) holds.*

Proof. We note that universal locally acyclicity is equivalent to dualizability in the sense of [Man26, Definition 7.5], compare [FS24, Proposition IV.2.19, Definition IV.2.22, Definition IV.2.31] with [Man26, Proposition 7.7, Corollary 7.8, Proposition 8.6]. By [Man26, Proposition 9.10] applied to

$$\mathrm{Igs}_{K^p,k} \xrightarrow{\bar{\pi}_{\mathrm{HT}}} \mathrm{Bun}_{G,\mu^{-1},k} \to \mathrm{Spd}\, k,$$

it suffices to check that $\bar{\pi}_{\mathrm{HT}}$ is cohomologically proper and that $\mathbb{W}$ is universally locally acyclic with respect to $\mathrm{Igs}_{K^p,k} \to \mathrm{Spd}\, k$. The properness follows from the properness of the Shimura variety, see Remark 5.2.3. The sheaf $\mathbb{W}$ is universally locally acyclic with respect to $\mathrm{Igs}_{K^p,k} \to \mathrm{Spd}\, k$ because the map is ℓ-cohomologically smooth by Proposition 5.1.9, and $\mathbb{W}$ is locally constant with finite projective stalks; for example, in [Man26, Proposition 7.7.(ii)] both sides are identified with $\pi_1^*\mathbb{W}^\vee \otimes \pi_2^*\mathbb{W}$ as the dualizing sheaf is trivial. □

5.2.7. *Igusa varieties and the Igusa sheaves.* For any element $b \in G(\breve{\mathbb{Q}}_p)$ with corresponding class $[b] \in B(G,\mu^{-1})$, we denote by i_b the locally closed immersion $i_b\colon \mathrm{Bun}_{G,k}^{[b]} \hookrightarrow \mathrm{Bun}_{G,k}$. Consider also the Newton stratum corresponding to $[b]$ in the Igusa stack $\mathrm{Igs}_{K^p}(\mathsf{G},\mathsf{X})$ with finite level away from p

$$\mathrm{Igs}_{K^p}^{[b]}(\mathsf{G},\mathsf{X}) := \mathrm{Igs}_{K^p} \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{[b]},$$

with $\overline{\pi}_b : \mathrm{Igs}_{K^p}^{[b]}(\mathsf{G},\mathsf{X}) \to \mathrm{Bun}_G^{[b]}$ the structure map. It follows from Lemma 5.1.1 that $\overline{\pi}_b$ is compactifiable, representable in locally spatial diamonds, and $\dim.\mathrm{trg}\,\overline{\pi}_b < \infty$. For $b : \mathrm{Spd}\,\overline{\mathbb{F}}_p \to \mathrm{Bun}_G^{[b]}$ we define the *v-sheaf Igusa variety*[14] $\mathrm{Ig}_{K^p}^{b,\mathrm{v}}(\mathsf{G},\mathsf{X})$ as the fiber product

$$\mathrm{Igs}_{K^p}^{[b]}(\mathsf{G},\mathsf{X}) \times_{\mathrm{Bun}_G^{[b]}} \mathrm{Spd}\,\overline{\mathbb{F}}_p.$$

If $x : \mathrm{Spa}\, C \to \mathrm{Spd}\,\overline{\mathbb{F}}_p \to \mathrm{Bun}_G^{[b]}$ is a morphism, then we write $\mathrm{Ig}_{K^p,C}^{b,\mathrm{v}}$ for the base-change of the v-sheaf Igusa variety to $\mathrm{Spa}\, C$. By pro-étale surjectivity of $\mathrm{Gr}_{G,\mu^{-1}} \to \mathrm{Bun}_{G,\mu^{-1}}$, the map x lifts to a $\mathrm{Spa}\, C$-point of $\mathrm{Gr}_{G,\mu^{-1}}$ and identifies $\mathrm{Ig}_{K^p,C}^{b,\mathrm{v}}$ with the fiber of $\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X}) \to \mathrm{Gr}_{G,\mu^{-1}}$. Let $\mathbb{W}$ be a standard Λ-local system.

Lemma 5.2.8. *In $\mathcal{D}(G_b(\mathbb{Q}_p),\Lambda)$, we have isomorphisms*

$$i_b^*\mathcal{F}_{c,\mathbb{W}} \simeq R\overline{\pi}_{b,!}\mathbb{W} \simeq R\Gamma_c(\mathrm{Ig}_{K^p,C}^{b,\mathrm{v}},\mathbb{W}),$$

where $\mathbb{W}$ denotes both the restriction of $\mathbb{W}$ to $\mathrm{Igs}_{K^p}^{[b]}(\mathsf{G},\mathsf{X})$ and its pullback to $\mathrm{Ig}_{K^p}^{b,\mathrm{v}}$.

[14]If $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ is not compact, then this conflicts with the notation in [DvHKZ24a, Section 4.4]. The v-sheaf Igusa varieties there are (the inverse limit over K^p) of $\mathrm{Ig}_{K^p}^{b,\circ,\mathrm{v}}(\mathsf{G},\mathsf{X}) := \mathrm{Igs}_{K^p}^{\circ} \times_{\mathrm{Bun}_G} \mathrm{Spd}\,\overline{\mathbb{F}}_p$

Proof. The first isomorphism follows from proper base change [Sch22, Proposition 22.19]. The second follows because $\mathrm{Ig}^{b,\mathrm{v}}_{K^p,C}$ is naturally identified with the fiber $\overline{\pi}_{\mathrm{HT}}^{-1}(x)$. ☐

Remark 5.2.9. If $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ is proper and Milne's axiom SV5 holds, then Conjecture 5.2.4(1) would also follow from a variant of Lemma 5.2.8; let us explain. Let $\mathcal{G}$ be a parahoric $\mathbb{Z}_p$-model for G and let $K_p = \mathcal{G}(\mathbb{Z}_p)$. When $p > 2$, by [DY25, Theorem A] there is a morphism

$$\mathscr{S}_K(\mathsf{G},\mathsf{X})^\diamond \to \mathrm{Sht}_{\mathcal{G},\mu}, \tag{5.2.1}$$

where $\mathscr{S}_K(\mathsf{G},\mathsf{X})$ is the $\mathcal{O}_E$-integral model for $\mathbf{Sh}_K(\mathsf{G},\mathsf{X})$ defined in [KPZ24], and $\mathrm{Sht}_{\mathcal{G},\mu}$ is the v-stack of $\mathcal{G}$-shtukas bounded by μ. Using (5.2.1), one can apply the construction of [HK25, Section 2.14] to obtain a *perfect Igusa variety* $\mathrm{Ig}^b_{K^p}$. We conjecture that, as in [DvHKZ24a, Lemma 8.5.3], one can identify $\mathrm{Ig}^{b,\mathrm{v}}_{K^p,C}$ with $(\mathrm{Ig}^b_{K^p})^\diamond_C$; this would follow from Conjecture 2.3.1. If so, the proof of [DvHKZ24a, Corollary 8.5.4] would imply that $\mathcal{F}$ is universally locally acyclic with respect to $\mathrm{Bun}_{G,\mu^{-1},k} \to \mathrm{Spd}\, k$.

Remark 5.2.10. Additionally in the compact case, part (2) of Conjecture 5.2.4 would follow from the arguments of [DvHKZ24a, Section 8.6] using the conjectural identification $\mathrm{Ig}^{b,\mathrm{v}}_{K^p,C} \simeq (\mathrm{Ig}^b_{K^p})_C$, but it would require additional inputs as well. Namely, the arguments of loc. cit. make use of a scheme-theoretic local model diagram in the sense of [DvHKZ24a, Conjecture 4.1.5] and the smoothness and affineness of central leaves in Newton strata. Neither of these results are known in the abelian-type case, but we expect (at least under mild assumptions) that they should be accessible.

5.3. **Cohomology of Shimura varieties.** The goal of this section is to give a formula for the (compactly supported) cohomology of $\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_C$ in terms of the action of a Hecke operator on the relative (compactly supported) cohomology of the Igusa stack over $\mathrm{Bun}_{G,k}$. This generalizes the main result of [DvHKZ24a, Section 8.4].

5.3.1. Let (G,X), E, v, E, G, and μ be as in Section 5.1. We also continue to fix a neat compact open subgroup $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$. Let $d = \langle 2\rho, \mu\rangle = \dim \mathsf{X}$. Let k be an algebraic closure of k_E, and let Λ be a $\mathbb{Z}_\ell$-algebra in which ℓ is nilpotent, that contains a square root of p. We choose such a square root and denote it by $\sqrt{p} \in \Lambda$. Let C be the completion of an algebraic closure of E, and denote by $\mathcal{O}_C$ the ring of integers of C. Fix an identification between k and the residue field of $\mathcal{O}_C$.

Lemma 5.3.2. *The morphism*

$$\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_K(\mathsf{G},\mathsf{X})$$

is a pro-étale torsor for the profinite group $U_p = K_p/N_p$, where $N_p = N_p(K) \subset K_p$ is the closed subgroup that is the image of the projection

$$\left(K \cap Z_{\mathsf{G}}(\mathbb{Q})^-\right) \to K_p.$$

Proof. This follows from the discussion in [KSZ21, Section 1.5.8], cf. [DY25, Section 4.1.2]. □

Definition 5.3.3. We say that K^p is *acceptable with respect to* K_p if the group $N_p(K_pK^p)$ is pro-p.

Note that if K^p is acceptable with respect to K_p, then it is also acceptable with respect to K_p' for any $K_p' \subset K_p$. The following lemma shows that there are many compact open subgroups K^p that are acceptable with respect to K_p.

Lemma 5.3.4. *For any compact open subgroup $K_p \subset G(\mathbb{Q}_p)$, there is a cofinal collection of compact open subgroups $U^p \subset \mathsf{G}(\mathbb{A}_f)$ such that the group $N_p(K_pU^p)$ is pro-p.*

Proof. Fix K_p as in the statement of the lemma and let $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ be compact open. It follows from the discussion in [KSZ21, Section 1.5.8] that K naturally surjects onto $\mathrm{Gal}(\mathbf{Sh}/\mathbf{Sh}_K)$ and similarly that

$$f_{K^p} : \{1\} \times K_p \subset K \to \mathrm{Gal}(\mathbf{Sh}/\mathbf{Sh}_K) \to \mathrm{Gal}(\mathbf{Sh}_{K^p}/\mathbf{Sh}_K)$$

is surjective. We are trying to show that f_{K^p} has pro-p kernel for a cofinal collection of K^p. Now, it is a direct consequence of the definitions that

$$\mathrm{Gal}(\mathbf{Sh}/\mathbf{Sh}_K) = \varprojlim_{U^p \subset K^p} \mathrm{Gal}(\mathbf{Sh}_{U^p}/\mathbf{Sh}_K).$$

Since $\{1\} \times K_p \subset K \to \mathrm{Gal}(\mathbf{Sh}/\mathbf{Sh}_K)$ is injective, see Lemma 2.3.4, and since kernels commute with inverse limits, we deduce that

$$\{1\} = \varprojlim_{U^p} \ker f_{U^p}.$$

In other words $\cap_{U^p} \ker f_{U^p} = \{1\}$, and thus there is a cofinal collection of U^p such that $\ker f_{U^p}$ is pro-p. Indeed, the group N_p contains a compact open pro-p group, since K_p does. □

5.3.5. Consider the commutative diagram

$$\begin{tikzcd}
\left[\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})/(\underline{G(\mathbb{Q}_p)} \times \phi^{\mathbb{Z}})\right] \arrow[d, "y"] & \left[\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})/(\underline{K_p} \times \phi^{\mathbb{Z}})\right] \arrow[l, "x"'] \arrow[r, "z"] \arrow[d, "h"] & \left[\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^{\diamondsuit}/\phi^{\mathbb{Z}}\right] \arrow[d, "f"] \\
\left[\mathrm{Spd}\, E/(\underline{G(\mathbb{Q}_p)} \times \phi^{\mathbb{Z}})\right] & \left[\mathrm{Spd}\, E/(\underline{K_p} \times \phi^{\mathbb{Z}})\right] \arrow[l, "b"] \arrow[r, "a"] & \left[\mathrm{Spd}\, E/\phi^{\mathbb{Z}}\right],
\end{tikzcd}$$

where the left square is Cartesian. Let $\mathbb{W}$ be a standard Λ-local system, and let us also write $\mathbb{W}$ for the canonical descent (in the sense of Section A.8) of $\mathbb{W}$ to the small v-stacks in the top row of the diagram. An important ingredient of our proof of Theorem 5.3.11 is an identification of $y_{k,!}\mathbb{W}$ and $Ry_{k,*}\mathbb{W}$, where k denotes the base change of the morphism along $\mathrm{Spd}\, k \to \mathrm{Spd}\, \mathbb{F}_p$. Since these objects are smooth representations of $G(\mathbb{Q}_p)$, it suffices to identify their K_p fixed vectors for a cofinal collection of compact open subgroups K_p. This is achieved by the following proposition.

Proposition 5.3.6. *If K^p is acceptable with respect to K_p, then there are natural isomorphisms*

$$a_{k,!}b_k^*y_{k,!}\mathbb{W} \to f_{k,!}\mathbb{W}$$
$$a_{k,!}b_k^*Ry_{k,*}\mathbb{W} \to Rf_{k,*}\mathbb{W}.$$

Lemma 5.3.7. *Let $c : X \to Y$ be an ℓ-fine morphism of small v-stacks that is a gerbe for $\underline{H}$, with H a p-adic Lie group. The morphism c is cohomologically proper, i.e., $Rc_* = c_!$. If H is pro-p, then the natural transformation $1 \to Rc_*c^*$ is an isomorphism.*

Proof. Because c is ℓ-fine, we can check that it is cohomologically proper after a v-cover. Thus we may assume that $X \simeq Y \times_{\operatorname{Spd}\overline{\mathbb{F}}_p} [\operatorname{Spd}\overline{\mathbb{F}}_p/\underline{H}]$, and then the cohomological properness is [Man26, Proposition 10.9, Lemma 9.8.(ii)]. Showing that $1 \to c_*c^* = c_!c^*$ is an isomorphism can also be done v-locally (by proper base change). Thus we may assume that $X \simeq Y \times_{\operatorname{Spd}\overline{\mathbb{F}}_p} [\operatorname{Spd}\overline{\mathbb{F}}_p/\underline{H}]$, and using proper base change again, we may further reduce to $c' : \operatorname{Spd}\overline{\mathbb{F}}_p \to \operatorname{Spd}\overline{\mathbb{F}}_p/\underline{H}$. Using [Man26, Lemma 10.3] to identify $c'_* = c'_!$ with continuous group cohomology, this follows from the fact that a pro-p group has trivial continuous group cohomology on Λ-modules. $\square$

Proof of Proposition 5.3.6. We first apply smooth (resp. proper) base-change to the left hand square of Section 5.3.5 to identify

$$b_k^*Ry_{k,*}\mathbb{W} = Rh_{k,*}\mathbb{W}$$
$$b_k^*y_{k,!}\mathbb{W} = h_{k,!}\mathbb{W}.$$

Next, recall that $K_p/N_p(K) = U_p(K)$, and consider the commutative diagram (the arrow is labeled c by abuse of notation because we will apply Lemma 5.3.7 to it)

$$\begin{tikzcd}
\left[\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^\diamondsuit/(\underline{K_p}\times\phi^{\mathbb{Z}})\right] \arrow[r, "z"] \arrow[d, "h"] & \left[\mathbf{Sh}_K(\mathsf{G},\mathsf{X})^\diamondsuit/\phi^{\mathbb{Z}}\right] \arrow[d, "g"] \arrow[dr, "f"] & \\
\left[\operatorname{Spd} E/(\underline{K_p}\times\phi^{\mathbb{Z}})\right] \arrow[r, "d"] & \left[\operatorname{Spd} E/(\underline{U_p(K)}\times\phi^{\mathbb{Z}})\right] \arrow[r, "c"] & \left[\operatorname{Spd} E/\phi^{\mathbb{Z}}\right],
\end{tikzcd}$$

where the square is Cartesian by Lemma 5.3.2. Noting that $a = c \circ d$, we identify (using the first part of Lemma 5.3.7)

$$a_{k,!}Rh_{k,*}\mathbb{W} = a_{k,*}Rh_{k,*}\mathbb{W} = Rc_{k,*}Rg_{k,*}Rz_{k,*}z_k^*\mathbb{W}$$
$$a_{k,!}h_{k,!}\mathbb{W} = c_{k,!}g_{k,!}z_{k,!}z_k^*\mathbb{W}.$$

By the second part of Lemma 5.3.7 and the fact that z is a gerbe for the pro-p (by assumption) group $N_p(K)$, we deduce that the natural map $\mathbb{W} \to Rz_{k,*}z_k^*\mathbb{W} = z_{k!}z_k^*\mathbb{W}$ is an isomorphism; since $f = c \circ g$, this allows us to conclude. $\square$

Remark 5.3.8. It follows from Proposition 5.3.6 that if K^p is acceptable with respect to K_p, then the (derived) Hecke algebra of level K_p acts on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_C^\diamondsuit, \mathbb{W})$ and $R\Gamma_c(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_C^\diamondsuit, \mathbb{W})$.

5.3.9. *Hecke operators.* Let $\mathrm{Div}^1_E = [\mathrm{Spd}\, E/\phi^{\mathbb{Z}}]$, where $\phi = \phi_{\mathrm{Spd}\, E}$ denotes the absolute Frobenius of $\mathrm{Spd}\, E$ (see [DvHKZ24a, Section 2.1.9]). By [FS24, Section III.3], the morphism $[\mathrm{Gr}_{G,\mu^{-1}}/\phi^{\mathbb{Z}}] \to \mathrm{Div}^1_E$ represents the functor sending $S \to \mathrm{Div}^1_E$ to the set of isomorphism classes of pairs consisting of a G-bundle $\mathscr{E}_1$ on X_S and a modification $\mathscr{E}^0 \mid_{(X_S)_E} \dashrightarrow \mathscr{E}_1 \mid_{(X_S)_E}$ from the trivial bundle $\mathscr{E}^0$ which is bounded by μ^{-1}. The functor sending such a modification to $\mathscr{E}^0$ defines a morphism

$$\tilde{h}_2 \colon [\mathrm{Gr}_{G,\mu^{-1}}/(\phi^{\mathbb{Z}} \times \underline{G(\mathbb{Q}_p)})] \to \mathrm{Bun}^{[1]}_G \times \mathrm{Div}^1_E,$$

see [DvHKZ24a, Section 8.4] for justification of the naming convention here. Additionally, the Beauville–Laszlo morphism

$$\mathrm{BL} \colon [\mathrm{Gr}_{G,\mu^{-1}}/(\phi^{\mathbb{Z}} \times \underline{G(\mathbb{Q}_p)})] \to \mathrm{Bun}_G$$

factors through $\mathrm{Bun}_{G,\mu^{-1}}$, and we denote the factorization $[\mathrm{Gr}_{G,\mu^{-1}}/(\phi^{\mathbb{Z}} \times \underline{G(\mathbb{Q}_p)})] \to \mathrm{Bun}_{G,\mu^{-1}}$ by g. As in [DvHKZ24a, Definition 8.4.7], we define the functor

$$\begin{aligned} T^{[1]}_\mu \colon \mathcal{D}(\mathrm{Bun}_{G,\mu^{-1},k}, \Lambda) &\to \mathcal{D}(\mathrm{Bun}^{[1]}_{G,k} \times_k \mathrm{Div}^1_{E,k}, \Lambda) \\ A &\mapsto R\tilde{h}_{2,k,*}(g_k^* A[d](d/2)). \end{aligned}$$

The operator $T^{[1]}_\mu$ is a version of a Hecke operator as defined in [FS24]; see [DvHKZ24a, Section 8.4.6] for the precise relation. We furthermore recall from [FS24, Proposition IV.7.1] that there is a fully faithful functor

$$\mathcal{D}(G(\mathbb{Q}_p), \Lambda)^{BW_E} \to \mathcal{D}(\mathrm{Bun}^{[1]}_{G,k} \times_k \mathrm{Div}^1_{E,k}, \Lambda),$$

and that $T^{[1]}_\mu$ lands in its essential image, see [FS24, Corollary IX.2.3].

5.3.10. Let Λ be a $\mathbb{Z}_\ell$-algebra in which ℓ is nilpotent, containing $\sqrt{p} \in \Lambda$ and let $\mathbb{W}$ be a standard local system. Recall that there is a natural Gal_E-equivariant isomorphism

$$R\Gamma(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_C, \mathbb{W}) \simeq \varinjlim_{K_p} R\Gamma(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_C, \mathbb{W}),$$

and we define (usually one defines compactly supported cohomology only for finite type separated schemes)

$$R\Gamma_c(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_C, \mathbb{W}) := \varinjlim_{K_p} R\Gamma_c(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_C, \mathbb{W}).$$

Both of these are naturally (complexes of) continuous representations of $G(\mathbb{Q}_p) \times \mathrm{Gal}_E$, such that the underlying (complex) of $G(\mathbb{Q}_p)$-representations consists of smooth representations. In other words, they can be considered as objects of $\mathcal{D}(G(\mathbb{Q}_p), \Lambda)^{BW_E}$. The following is the main result of this section, cf. [DvHKZ24a, Theorem 8.4.10].

Theorem 5.3.11. *For a standard Λ-local system $\mathbb{W}$, there are isomorphisms*

$$\begin{aligned} R\Gamma_c(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_C, \mathbb{W}) &\simeq T^{[1]}_\mu(\mathcal{F}_{\mathbb{W},c}[-d])(-d/2), \\ R\Gamma(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_C, \mathbb{W}) &\simeq T^{[1]}_\mu(\mathcal{F}_{\mathbb{W}}[-d])(-d/2) \end{aligned}$$

in $\mathcal{D}(G(\mathbb{Q}_p), \Lambda)^{BW_E}$.

Remark 5.3.12. Our proof of Theorem 5.3.11 below (or the proof of [DvHKZ24a, Theorem 8.4.10]) can also be used to show a version of Theorem 5.3.11 with $\mathcal{F}^\circ_{\mathbb{W}}$ instead of $\mathcal{F}_{\mathbb{W}}$. The main difficulty is to generalize [DvHKZ24a, Proposition 8.4.12], comparing the cohomology of the good reduction locus with the cohomology of the whole Shimura variety, to the case of abelian type Shimura varieties. This is now possible thanks to the recent work of Wu, see [Wu25, Proposition 5.21], which replaces the use of [LS18, Corollary 5.20] in the proof of [DvHKZ24a, Proposition 8.4.12].

Proof of Theorem 5.3.11. From the Cartesian diagram (4.5.1) we obtain a 2-commutative diagram, where we relabel the arrows in the interest of brevity:

$$\begin{tikzcd}
\left[\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^\diamondsuit/(\underline{G(\mathbb{Q}_p)}\times\phi^{\mathbb{Z}})\right] \arrow[r, "\pi"] \arrow[d, "\tilde{g}"] & \left[\mathrm{Gr}_{G,\mu^{-1}}/(\underline{G(\mathbb{Q}_p)}\times\phi^{\mathbb{Z}})\right] \arrow[d, "g"] \\
\mathrm{Igs}_{K^p}(\mathsf{G},\mathsf{X}) \arrow[r, "\overline{\pi}"] & \mathrm{Bun}_{G,\mu^{-1}}.
\end{tikzcd}$$

As in the proof of [DvHKZ24a, Theorem 8.4.10], we see that $\tilde{h}_2 \circ \pi$ is the structure morphism

$$y : [\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})^\diamondsuit/(\phi^{\mathbb{Z}}\times\underline{G(\mathbb{Q}_p)})] \to [\mathrm{Spd}\, E/(\phi^{\mathbb{Z}}\times\underline{G(\mathbb{Q}_p)})].$$

By definition we have

$$\begin{aligned}
T^{[1]}_\mu(\mathcal{F}_{\mathbb{W},c}[-d])(-d/2) &\simeq R\tilde{h}_{2,k,*}g_k^*\overline{\pi}_{k,!}\mathbb{W} \\
T^{[1]}_\mu(\mathcal{F}_{\mathbb{W}}[-d])(-d/2) &\simeq R\tilde{h}_{2,k,*}g_k^*R\overline{\pi}_{k,*}\mathbb{W}
\end{aligned}$$

By [SW20, Proposition 20.2.3], the map $\mathrm{Gr}_{G,\mu^{-1}} \to \mathrm{Spd}\, E$ is proper, and therefore $\tilde{h}_2$ is proper as well. Thus by applying proper (resp. smooth) base change, we obtain

$$\begin{aligned}
T^{[1]}_\mu(\mathcal{F}_{\mathbb{W},c}[-d])(-d/2) &\simeq \tilde{h}_{2,k,!}\pi_{k,!}\mathbb{W} = y_{k,!}\mathbb{W} \\
T^{[1]}_\mu(\mathcal{F}_{\mathbb{W}}[-d])(-d/2) &\simeq R\tilde{h}_{2,k,*}R\pi_{k,*}\mathbb{W} = Ry_{k,*}\mathbb{W}.
\end{aligned}$$

Since these are (complexes of) smooth representations of $G(\mathbb{Q}_p)$, we can recover them from the colimit of their K_p-invariants. Applying Proposition 5.3.6 then gives us $G(\mathbb{Q}_p)$-equivariant isomorphisms in $\mathcal{D}(\Lambda)^{BW_E}$

$$\begin{aligned}
T^{[1]}_\mu(\mathcal{F}_{\mathbb{W},c}[-d])(-d/2) &\simeq \varinjlim_{K_p} f_{K_p,k,!}\mathbb{W}, \\
T^{[1]}_\mu(\mathcal{F}_{\mathbb{W}}[-d])(-d/2) &\simeq \varinjlim_{K_p} Rf_{K_p,k,*}\mathbb{W},
\end{aligned}$$

where f_{K_p} is the structure map

$$\left[\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})^\diamondsuit/(\underline{K_p}\times\phi^{\mathbb{Z}})\right] \to \left[\mathrm{Spd}\, E/\phi^{\mathbb{Z}}\right].$$

By Proposition A.14 and comparison theorems between the étale cohomology of an algebraic variety and its analytification, see Lemma A.15 and [Sch22, Lemma 15.6, Proposition 27.5], for each K_p, there are canonical W_E-equivariant identifications

$$\begin{aligned} Rf_{K_p,k,*}\mathbb{W} &\simeq R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_C, \mathbb{W}),\\ f_{K_p,k,!}\mathbb{W} &\simeq R\Gamma_c(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_C, \mathbb{W}),\end{aligned}$$

finishing the proof. □

5.4. **A product formula.** In this section we prove a Mantovan-style product formula, following [DvHKZ24a, Section 8.5]. Let Λ be a $\mathbb{Z}_\ell$-algebra in which ℓ is nilpotent containing a fixed square root of p. Let k be an algebraic closure of the residue field of E and base-change everything to k as before.

5.4.1. Before we can state the Mantovan product formula, we need to establish a bit more notation. First fix a left-invariant Haar measure on $G_b(\mathbb{Q}_p)$ and write $\mathcal{H}(G_b)$ for the algebra $C_c(G_b(\mathbb{Q}_p), \Lambda)$ of compactly supported locally constant Λ-valued functions on $G_b(\mathbb{Q}_p)$. Additionally, define the local Shimura variety at infinite level attached to (G, b, μ) as the fiber product

$$\begin{tikzcd} \mathcal{M}_{G,b,\mu,\infty} \arrow[r] \arrow[d] & \mathrm{Gr}^{[b]}_{G,\mu^{-1}} \arrow[d, "\mathrm{BL}"] \\ \mathrm{Spd}\, k \arrow[r, "b"] & \mathrm{Bun}^{[b]}_G. \end{tikzcd} \tag{5.4.1}$$

We note that the top horizontal arrow in this diagram is a $\tilde{G}_b$-torsor, and the action of $\underline{G(\mathbb{Q}_p)}$ on $\mathrm{Gr}^{[b]}_{G,\mu^{-1}}$ lifts to an action on $\mathcal{M}_{G,b,\mu,\infty}$ commuting with the $\tilde{G}_b$-action.

By [HI25, Proposition 3.15], there is a character δ_b of $G_b(\mathbb{Q}_p)$ such that the dualizing complex of $\mathrm{Bun}^{[b]}_{G,k}$ is given by $\delta_b^{-1}[-2d_b]$, where $d_b = \langle 2\rho, \nu_b\rangle$ (here, as usual, δ_b is viewed as a sheaf on $\mathrm{Bun}^{[b]}_{G,k}$ using the equivalence $\mathcal{D}(\mathrm{Bun}^{[b]}_{G,k}, \Lambda) \simeq \mathcal{D}(G_b(\mathbb{Q}_p), \Lambda)$ as in [FS24, Theorem I.5.1.(ii)]). We define $R\Gamma_c(\mathcal{M}_{G,b,\mu,\infty}, \delta_b)$ to be the exceptional pushforward of δ_b along the structure map $\mathcal{M}_{G,b,\mu,\infty} \to \mathrm{Spd}\, C$, where we view δ_b as a sheaf on $\mathcal{M}_{G,b,\mu,\infty}$ by pulling back along either composition in the diagram (5.4.1). As in [DvHKZ24a, Section 8.5], $\mathcal{M}_{G,b,\mu,\infty}$ admits a descent to $\mathrm{Div}^1_{E,k}$, and it follows that $R\Gamma_c(\mathcal{M}_{G,b,\mu,\infty}, \delta_b)$ is equipped with an action of $G_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times W_E$.

Theorem 5.4.2. *There exists a filtration on $R\Gamma_c(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_{\overline{E}}, \mathbb{W})$ of complexes of smooth representations of $G(\mathbb{Q}_p) \times W_E$, whose graded pieces are given by*

$$R\Gamma_c(\mathrm{Ig}^{b,\mathrm{v}}_{K^p,C}, \mathbb{W})^{\mathrm{op}} \otimes^{\mathbb{L}}_{\mathcal{H}(G_b)} R\Gamma_c(\mathcal{M}_{G,b,\mu,\infty}, \delta_b)[2d_b].$$

Proof. The arguments in the proof of [DvHKZ24a, Theorem 8.5.7] show that $R\Gamma_c(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_{\overline{E}}, \mathbb{W})$ admits a filtration whose graded pieces are given by

$$R\overline{\pi}_{b,!}\mathbb{W} \otimes^{\mathbb{L}}_{\mathcal{H}(G_b)} R\Gamma_c(\mathcal{M}_{G,b,\mu,\infty}, \delta_b)[2d_b].$$

We then conclude using Lemma 5.2.8. □

5.5. **Compatibility with the Fargues–Scholze correspondence.** We continue to let (G, X), E, v, E, G, k, and μ be as in Section 5.1. Let us assume that Λ is a $\mathbb{Z}_\ell$-algebra containing a square root of p in which ℓ is nilpotent. We fix a compact open subgroup $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$, and for any compact open subgroup $K_p \subset G(\mathbb{Q}_p)$, we write $K = K_p K^p$. Moreover, we let $d = \langle 2\rho, \mu \rangle$ be the dimension of the Shimura variety. In this section we study the Igusa sheaves using the spectral action of [FS24]. Our exposition here largely follows [DvHKZ24a, Section 9], and we refer the reader there for detailed explanations of the arguments which are here only briefly summarized.

5.5.1. To state the theorem in this section we need to recall a bit of notation. Let $\widehat{G}$ be a dual group of G over $\mathbb{Z}_\ell(\sqrt{p})$, equipped with an action of $W_{\mathbb{Q}_p}$ restricted from an action of $\mathrm{Gal}_{\mathbb{Q}_p}$. As in [BG14, Section 2.1], define the L-group ${}^L G$ to be $\widehat{G} \rtimes W_{\mathbb{Q}_p}$. Associated with μ is a flat perverse sheaf $\mathcal{S}_\mu$, which lies in the Satake category for the local Hecke stack $\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1}$, see [DvHKZ24a, Section 8.4.1] for the precise definition. Applying the fiber functor F defined in [FS24, Definition / Proposition IV.7.10] to $\mathcal{S}_\mu$, we obtain a $W_{\mathbb{Q}_p}$-equivariant algebraic representation of the dual group $\widehat{G}$, which formally corresponds to a representation r_μ of ${}^L G = \widehat{G} \rtimes W_{\mathbb{Q}_p}$. Note here we are applying [FS24, Theorem VI.11.1] to identify the dual group $\widehat{G}$ with the Tannaka group $\check{G}$ of the Satake category $\mathrm{Sat}(\mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^1}, \Lambda)$; this uses our choice of $\sqrt{p} \in \mathbb{Z}_\ell(\sqrt{p})$.

Denote by Par_G the stack of L-parameters over $\mathbb{Z}_\ell(\sqrt{p})$ as in [DHKM25], [Zhu25b], and [FS24], and let $\mathrm{Perf}(\mathrm{Par}_G)$ be the ∞-category of perfect complexes on Par_G. By [FS24, Theorem X.0.2, Theorem V.4.1], when ℓ is coprime to the order of $\pi_0(Z(G))$, there is a natural Λ-linear action (the spectral action) of $\mathrm{Perf}(\mathrm{Par}_G)$ on $\mathcal{D}(\mathrm{Bun}_{G,k}, \Lambda)$, which preserves the compact objects. If $\mathcal{V}$ is a perfect complex on Par_G, we denote the spectral action of $\mathcal{V}$ on $\mathcal{D}(\mathrm{Bun}_{G,k}, \Lambda)$ by $\mathcal{V} * (-)$.

5.5.2. When $\mathcal{V}$ is defined from an algebraic representation V of $\widehat{G}$, the spectral action is naturally identified with the Hecke operator T_V defined in [FS24, Chapter IX], see [DvHKZ24a, Example 9.1.2]. In particular, the representation r_μ determines a vector bundle $\mathcal{V}_\mu$ on Par_G, and the spectral action of $\mathcal{V}_\mu$ recovers the action of the Hecke operator T_μ. We write $\pi_0 \mathrm{End}_{\mathrm{Par}_G}(\mathcal{V}_\mu)$ for the ring of endomorphisms of $\mathcal{V}_\mu$, considered as a vector bundle on Par_G (as opposed to the endomorphisms in the category of perfect complexes).

For an algebraic representation V and corresponding vector bundle $\mathcal{V}$ on Par_G, via the action of $\mathrm{End}_{\mathrm{Par}_G}(\mathcal{V})$ the objects $\mathcal{V} * A$ for A in $\mathcal{D}(\mathrm{Bun}_{G,k}, \Lambda)$ inherit an action of W_E, which we call the *spectral W_E-action*. This action is compatible with the natural action of W_E on the Hecke operators T_V under the identification $\mathcal{V} * (-) = T_V$, see [DvHKZ24a, Remark 9.1.3].

5.5.3. In the rest of this section, we will implicitly base-change the above objects along $\mathbb{Z}_\ell(\sqrt{p}) \to \Lambda$, using our fixed $\sqrt{p} \in \Lambda$.

5.5.4. If ℓ is coprime to the order of $\pi_0(Z(G))$, then by [FS24, Corollary IX.0.3], there is a morphism

$$\Psi_G\colon \mathcal{Z}^{\mathrm{spec}}(G,\Lambda)\to \mathcal{Z}(G(\mathbb{Q}_p),\Lambda), \tag{5.5.1}$$

where $\mathcal{Z}^{\mathrm{spec}}(G,\Lambda)=\Gamma(\mathrm{Par}_G,\mathcal{O})$ is the spectral Bernstein center [FS24, Definition IX.0.2] and $\mathcal{Z}(G(\mathbb{Q}_p),\Lambda)$ is the (usual) Bernstein center of the abelian category of smooth $G(\mathbb{Q}_p)$-representations on Λ-modules.

Finally, let us denote by $j_\mu\colon \mathrm{Bun}_{G,\mu^{-1}}\hookrightarrow \mathrm{Bun}_G$ the open immersion of the μ^{-1}-admissible locus and by $i_1\colon \mathrm{Bun}_G^{[1]}\hookrightarrow \mathrm{Bun}_G$ the open immersion of the Newton stratum corresponding to $1\in G(\breve{\mathbb{Q}}_p)$.

Proposition 5.5.5. *Assume that ℓ is coprime to the order of $\pi_0(Z(G))$, and let $\mathbb{W}$ be a standard Λ-local system. Then there is an isomorphism*

$$R\Gamma(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})\simeq i_{1,k}^*\left(\mathcal{V}_\mu * (j_{\mu,k,!}\mathcal{F}_{\mathbb{W}}[-d])(-\frac{d}{2})\right)$$

in $\mathcal{D}(G(\mathbb{Q}_p),\Lambda)^{BW_E}\simeq \mathcal{D}(\mathrm{Bun}_{G,k}^{[1]},\Lambda)^{BW_E}$. Moreover, if K_p is a compact open subgroup of $\mathsf{G}(\mathbb{Q}_p)$ such that K^p is acceptable with respect to K_p, then there is a map of Λ-algebras

$$\pi_0\,\mathrm{End}_{\mathrm{Par}_G}(\mathcal{V}_\mu)\to \mathrm{End}_{\mathcal{D}(G(\mathbb{Q}_p),\Lambda)}(R\Gamma(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})).$$

Proof. The isomorphism follows from Theorem 5.3.11. Moreover, by Proposition 5.3.6, when K^p is acceptable with respect to K_p, the complex $R\Gamma(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})$ is identified with the derived K_p-invariants of $R\Gamma(\mathbf{Sh}_{K^p}(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})$. In that case the existence of the map of Λ-algebras is proven exactly as in [DvHKZ24a, Theorem 9.1.4]. $\square$

5.5.6. Suppose K^p is acceptable with respect to K_p, and let $K=K_pK^p$. Then from Proposition 5.5.5, we obtain an action of W_E on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})$ for any W and Λ as in the statement of the theorem. We refer to this as the spectral W_E-action on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})$. If Λ is instead the ring of integers in a finite extension of $\mathbb{Q}_\ell$, then we can consider local systems $\mathbb{W}$ constructed from a continuous action of K^p on a finite free Λ-module W, which factors through the image of K^p in $\mathsf{G}^c(\mathbb{A}_f^p)$; we will call such local systems standard Λ-local systems. For such a standard Λ-local system, if we fix $\sqrt{p}\in\Lambda$, then we similarly obtain a W_E-action on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})$ (again called the spectral W_E-action) by taking a limit of the actions on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W}/\ell^n\mathbb{W})$ for each n.

Theorem 5.5.7. *Suppose Λ is a $\mathbb{Z}_\ell$-algebra containing a fixed square root of p which is either ℓ-torsion or the ring of integers in a finite extension of $\mathbb{Q}_\ell$, and let $\mathbb{W}$ be a standard Λ-local system. Assume that ℓ is coprime to the order of $\pi_0(Z(G))$. If K^p is acceptable with respect to K_p, then $\pi_0\,\mathrm{End}_{\mathrm{Par}_G}(\mathcal{V}_\mu)$ acts on $R\Gamma(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})$. Moreover, the spectral and usual W_E-action on $R\Gamma(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_{\overline{E}},\mathbb{W})(\frac{d}{2})$ agree*[15]*,*

[15] The Tate twist by $(d/2)$ was erroneously omitted in [DvHKZ24a, Theorem 9.1.4 and Corollary 9.1.5 of v1].

and the action of $\mathcal{Z}^{\mathrm{spec}}(G,\Lambda) \subset \pi_0 \operatorname{End}_{\mathrm{Par}_G}(\mathcal{V}_\mu)$ on $R\Gamma(\mathbf{Sh}_{K_pK^p}(\mathsf{G},\mathsf{X})_{\overline{E}}, \mathbb{W})$ factors via Ψ_G through the natural action of $\mathcal{Z}(G(\mathbb{Q}_p),\Lambda)$.

Proof. The theorem follows from Proposition 5.5.5 by the arguments in the proofs of [DvHKZ24a, Theorem 9.1.4, Corollary 9.1.5]. □

Remark 5.5.8. Let us denote the (possibly) non-commutative Λ-algebra $\pi_0 \operatorname{End}_{\mathrm{Par}_G}(\mathcal{V}_\mu)$ by $E_{v,\mu}$. For each place v of E not above ℓ, the above result gives an algebra homomorphism $E_{v,\mu} \to \operatorname{End}_\Lambda(R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{W}))$. Then by varying v, we get an algebra homomorphism from the free product $*_{v\nmid\ell} E_{v,\mu} \to \operatorname{End}_\Lambda(R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{W}))$. The image of this map contains the (global) unramified Hecke algebra, and should encode certain (global) Galois theoretic information, since the vector bundles $\mathcal{V}_\mu$ carry Weil group actions. Though it seems hard to describe this image explicitly.

5.5.9. In the remainder of this subsection we assume Λ is the ring of integers in a finite extension of $\mathbb{Q}_\ell$ and that Λ contains a fixed square root of p. We fix $K = K_pK^p$ as before with K^p acceptable with respect to K_p. We write $\mathcal{H}_{K_p}$ for the Hecke algebra $\Lambda[G(\mathbb{Q}_p) /\!\!/ K_p]$ of level K_p with coefficients in Λ, and fix a standard Λ-local system $\mathbb{W}$. Since K^p is acceptable with respect to K_p, the complex $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}}, \mathbb{W})$ carries a natural right action by $\mathcal{H}_{K_p}$, see Remark 5.3.8. Let $\mathcal{Z}_{K_p}$ denote the center of $\mathcal{H}_{K_p}$, and suppose $\chi\colon \mathcal{Z}_{K_p} \to L$ is a character, where L is either $\overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$. For any i, define[16]

$$W^i(\chi) := H^i(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}}, \mathbb{W})(d/2) \otimes_{\mathcal{Z}_{K_p},\chi} L.$$

The following theorem describes a compatibility between the Fargues–Scholze local Langlands correspondence and the cohomology of Shimura varieties. Recall that using (5.5.1) and [FS24, Proposition VIII.3.2], we can associate to χ a semisimple L-parameter $\phi_\chi^{\mathrm{FS}} : W_{\mathbb{Q}_p} \to {}^LG(L)$.

Theorem 5.5.10. *Suppose ℓ is coprime to the order of $\pi_0(Z(G))$. Then each irreducible L-linear W_E-representation occurring as a subquotient of $W^i(\chi)$ also occurs as a subquotient of $r_\mu \circ \phi_\chi^{\mathrm{FS}}\big|_{W_E}$.*

Proof. This follows from the same proof as that of [DvHKZ24a, Theorem 9.2.1], using Theorem 5.5.7 to construct the commutative diagrams therein. □

5.6. **Hecke polynomials and the Eichler–Shimura relation.** With the results of the previous section in hand, we can now derive an Eichler–Shimura congruence relation for Shimura varieties of abelian type. For this we follow [DvHKZ24a, Section 9.4] and [vdH24, Section 5].

Let Λ be the ring of integers in a finite extension of $\mathbb{Q}_\ell$ containing a fixed square root of p. Fix an algebraic representation V of $\widehat{G}$, and assume V is of rank n over Λ. Suppose that it extends to a representation $r_V : \widehat{G} \rtimes W_E \to \mathrm{GL}(V)$ for some finite extension E over $\mathbb{Q}_p$, and let ϕ_E^{univ} be the universal 1-cocycle on $Z^1(W_{\mathbb{Q}_p}, \widehat{G})$

[16] The Tate twist by $(d/2)$ was erroneously omitted in [DvHKZ24a, Section 1.2, Section 9.2 of v1].

restricted to W_E. Then for any $\gamma \in W_E$ we obtain an element $(r_V \circ \phi_E^{\mathrm{univ}})(\gamma)$ of $\pi_0 \operatorname{End}_{\operatorname{Par}_G}(\mathcal{V})$, where as before $\mathcal{V}$ is the vector bundle on Par_G corresponding to V.

5.6.1. Following [DvHKZ24a, Definition 9.3.2], for $\gamma \in W_E$, we define the spectral Hecke polynomial as

$$H^{\mathrm{spec}}_{G,V,\gamma}(X) = \det(X - (r_V \circ \phi_E^{\mathrm{univ}})(\gamma)) = \sum_{i=0}^{n} (-1)^i S_{\wedge^{n-i}V,\gamma} X^i \in \mathcal{Z}^{\mathrm{spec}}(G, \Lambda).$$

Here $S_{\wedge^i V,\gamma} \in \mathcal{Z}^{\mathrm{spec}}(G, \Lambda)$ is an excursion operator whose attached function sends an L-parameter ϕ to the trace of $\wedge^i(r_V \circ \phi)(\gamma)$, see [DvHKZ24a, Section 9.3] for details. By [DvHKZ24a, Corollary 9.3.3], there is a $\mathcal{Z}^{\mathrm{spec}}(G, \Lambda)$-algebra homomorphism

$$\mathcal{Z}^{\mathrm{spec}}(G, \Lambda)[X]/(H^{\mathrm{spec}}_{G,V,\gamma}(X)) \to \pi_0 \operatorname{End}_{\operatorname{Par}_G}(\mathcal{V}) \tag{5.6.1}$$

given by mapping X to $(r_V \circ \phi_E^{\mathrm{univ}})(\gamma)$.

5.6.2. Let $H_{G,V,\gamma}(X)$ denote the image of $H^{\mathrm{spec}}_{G,V,\gamma}$ under the map Ψ_G (see (5.5.1)). We have the following proposition, which follows exactly as [DvHKZ24a, Corollary 9.3.5].

Proposition 5.6.3. *Let ℓ be coprime to the order of $\pi_0(Z(G))$. Let $K = K_p K^p$ be a compact open subgroup such that K^p is acceptable with respect to K_p. Then for every element $\gamma \in W_E$ and every standard Λ-local system $\mathbb{W}$ the endomorphism $H_{G,V_\mu,\gamma}(\gamma)$ acts as zero on $R\Gamma(\mathbf{Sh}_K(\mathsf{G},\mathsf{X})_{\overline{E}}, \mathbb{W})(\frac{d}{2})$.*

Proof. Given the homomorphism (5.6.1), this is immediate from Theorem 5.5.7 and Proposition 5.3.6. □

Remark 5.6.4. If γ is a lift of Frobenius, then the Hecke polynomial $H_{G,V_\mu,\gamma}$ is independent of ℓ. More precisely, it follows from [Sch25, Theorem 6.2, proof of Corollary 6.3] that it lies inside $\mathcal{Z}(G(\mathbb{Q}_p), \mathbb{Z}[\sqrt{p}, \frac{1}{p|\pi_0(Z_{\mathsf{G}})|}]) \subset \mathcal{Z}(G(\mathbb{Q}_p), \Lambda)$. Here the inclusion of Bernstein centers is described in [DHKM24b, Lemma 3.2].

5.6.5. Now let us fix a special parahoric subgroup $K_p \subset G(\mathbb{Q}_p)$ with $\mathbb{Z}_p$-model $\mathcal{G}$, and let $K'_p \subset K_p$ be an Iwahori subgroup with $\mathbb{Z}_p$-model $\mathcal{I}$. Write $K' = K'_p K^p$. Fix also a quasi-split inner form G^* of G, along with a very special parahoric integral model $\mathcal{G}^*$. Let $\mathcal{H}_{\mathcal{G}}$, $\mathcal{H}_{\mathcal{I}}$, and $\mathcal{H}_{\mathcal{G}^*}$ denote the Λ-linear Hecke algebras for $\mathcal{G}$, $\mathcal{I}$, and $\mathcal{G}^*$, respectively.

Let q be the cardinality of the residue field of E, and let $I = I_{\mathbb{Q}_p}$ denote the inertia subgroup of $W_{\mathbb{Q}_p}$. Fix (arithmetic) Frobenius lifts $\sigma \in W_{\mathbb{Q}_p}$ and $\sigma_E \in W_E$. Because G^* is quasi-split, [vdH24, Proposition 2.2] provides an isomorphism between $\mathcal{H}_{\mathcal{G}^*}$ and the algebra of $\widehat{G}^I$-invariant functions on $\widehat{G}^I q^{-1}\sigma$. Following [vdH24, Section 5], using this isomorphism we define an auxiliary renormalized Hecke polynomial in $\mathcal{H}_{\mathcal{G}^*}[X]$ by

$$H^*_\mu = \det(X - q^{\frac{d}{2}} r_\mu(g, \sigma_E)).$$

Following [vdH24, Section 3], we write

$$\tilde{t} \colon \mathcal{H}_{\mathcal{G}^*} \to \mathcal{H}_{\mathcal{G}}$$

for the normalized transfer homomorphism between $\mathcal{H}_{\mathcal{G}^*}$ and $\mathcal{H}_{\mathcal{G}}$. Then the renormalized Hecke polynomial $H_\mu \in \mathcal{H}_{\mathcal{G}}[X]$ is defined to be the image of H_μ^* under $\tilde{t}$, i.e.,

$$H_\mu = \tilde{t}(H_\mu^*).$$

We can now state our main theorem of this section. Let K_p' be as above.

Theorem 5.6.6. *Assume that ℓ be coprime to the order of $\pi_0(Z(G))$. Let $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ be a compact open subgroup such that K^p is acceptable with respect to K_p', set $K' = K_p' K^p$, and let $\mathbb{W}$ be a standard Λ-local system. Then the inertia subgroup I_E of W_E acts unipotently on $H^i(\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X})_{\overline{E}}, \mathbb{W})$ for all i. Moreover, the action of any Frobenius lift $\sigma_E \in W_E$ on $R\Gamma(\mathbf{Sh}_{K'}(\mathsf{G}, \mathsf{X})_{\overline{E}}, \mathbb{W})$ satisfies $H_\mu(\sigma_E) = 0$.*

Proof. This follows from Proposition 5.6.3 by the arguments in the proof of [vdH24, Theorem 5.4], which comprise a generalization of the arguments in the proof of [DvHKZ24a, Theorem 9.4.11]. □

5.7. **Partial Frobenii and their Eichler–Shimura relations.** In this subsection, we show that when an abelian type Shimura variety comes from restriction of scalars from a totally real field, then at split primes its cohomology satisfies the local plectic conjecture of Nekovár–Scholl.[17] If we take the level at p to be hyperspecial, this proves the existence of partial Frobenius operators each satisfying an Eichler–Shimura relation; this generalizes a result of Lee [Lee22, Theorem 1.0.4]. Let Λ be a $\mathbb{Z}_\ell$-algebra in which ℓ is nilpotent, containing a fixed $\sqrt{p} \in \Lambda$.

5.7.1. More precisely, we let (G, X) be an abelian type Shimura datum, such that $\mathsf{G} = \mathrm{Res}_{\mathbb{Q}}^{\mathsf{F}} \mathsf{H}$ for some connected reductive group H over a totally real number field F of degree r over $\mathbb{Q}$. We assume that p splits completely in F, and let $\mathfrak{p}_i \mid p$, $i = 1, \dots, r$ be primes of F above p. Denote by $F_i \simeq \mathbb{Q}_p$ the completion of F at $\mathfrak{p}_i$. Then $G = \mathsf{G}_{\mathbb{Q}_p}$ is isomorphic to $\prod_{i=1}^r H_i$, where $H_i = \mathsf{H} \times_{\mathsf{F}} F_i$. We can similarly identify

$$\mathrm{Aut}_{\mathsf{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p}(\mathsf{F} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p) \simeq \prod_{i=1}^r \mathrm{Gal}_{\mathbb{Q}_p},$$

which contains a "diagonal" copy of $\mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathsf{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p}(\mathsf{F} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p)$.

5.7.2. Let E be the reflex field of (G, X) as before. Choose an isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, which induces a p-adic place v of E. Let $E = \mathsf{E}_v$ be the completion. The Hodge cocharacter μ can be viewed as a cocharacter of G via ι_p and it decomposes as a product $\mu = \prod_{i=1}^r \mu_i$, where μ_i is a cocharacter of H_i. For each i, we let E_i be the reflex field of μ_i. The group $\prod_{i=1}^r \mathrm{Gal}_{\mathbb{Q}_p}$ naturally acts on the space of conjugacy classes of cocharacters of G, and its stabilizer is given by $\prod_{i=1}^r \mathrm{Gal}_{E_i}$. Intersecting with the diagonal $\mathrm{Gal}_{\mathbb{Q}_p}$ gives the stabilizer Gal_E, and in particular there is a natural injective map $\mathrm{Gal}_E \to \prod_{i=1}^r \mathrm{Gal}_{E_i}$. We consider the induced embedding $W_E \to \prod_{i=1}^r W_{E_i}$, restricting along which defines a functor between the

[17]The local plectic conjecture is most interesting at primes that are not split. For such primes, it will be proved in upcoming work of Feng–Tamiozzo–Zhang.

corresponding derived categories of continuous representations $\mathcal{D}(\prod_{i=1}^r W_{E_i}, \Lambda) \to \mathcal{D}(W_E, \Lambda)$. We have the following result, which is a special case of the local version of Nekovár–Scholl's plectic conjecture.

Theorem 5.7.3. *Let $K = K_p K^p \subset \mathsf{G}(\mathbb{A}_f)$ be a neat compact open subgroup such that K^p is acceptable with respect to K_p and let $\mathbb{W}$ be a standard Λ-local system. If ℓ is coprime to the order of $\pi_0(Z(G))$, then the cohomology complex*

$$R\Gamma(\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})_{\overline{E}}, \mathbb{W})[d](d/2) \in \mathcal{D}(W_E, \Lambda)$$

lifts canonically to an object in $\mathcal{D}(\prod_i W_{E_i}, \Lambda)$, compatibly with the Hecke actions.

Proof. In our situation, the stack of L-parameters for G decomposes as a product

$$\mathrm{Par}_{G_{\mathbb{Q}_p}} \simeq \prod_{i=1}^r \mathrm{Par}_{H_{i,F_i}}.$$

Under this isomorphism, we have, accordingly, a decomposition $\mathcal{V}_\mu \simeq \boxtimes_{i=1}^r \mathcal{V}_{\mu_i}$, where $\mathcal{V}_{\mu_i}$ is a vector bundle on Par_{H_i,F_i} corresponding to the tilting representations of $\widehat{H}_i$ labeled by μ_i. Each $\mathcal{V}_{\mu_i}$ carries an action of W_{E_i} and the decomposition $\mathcal{V}_\mu \simeq \boxtimes_{i=1}^r \mathcal{V}_{\mu_i}$ is equivariant for the W_E-action on the left hand side and the $\prod_i W_{E_i}$-action on the right hand side with respect to the diagonal embedding $W_E \to \prod_i W_{E_i}$. The desired result now follows from Theorem 5.3.11 and Proposition 5.3.6. □

Remark 5.7.4. For $p > 2$ and $G_{\mathbb{Q}_p}$ unramified, this is proved with $\overline{\mathbb{Z}}_\ell$ coefficients for the basic part of the cohomology of the Shimura variety by Li-Huerta [LH22, Theorem A].

5.7.5. Now let K_p be a hyperspecial parahoric subgroup of $G(\mathbb{Q}_p)$ with integral model $\mathcal{G}$, and let $\mathcal{H}_\mathcal{G}$ be the Hecke algebra for K_p with Λ-coefficients as before. Note that we can write $\mathcal{G} = \prod_i \mathcal{G}_i$ and $\mathcal{H}_\mathcal{G} = \otimes_i \mathcal{H}_{\mathcal{G}_i}$. Let k_E be the residue field of E, with cardinality q, and denote by $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{k}_E/k_E)$ the arithmetic Frobenius. Fix a neat compact open subgroup $K^p \subset \mathsf{G}(\mathbb{A}_f^p)$ which is acceptable with respect to K_p. Recall that $R\Gamma(\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})_{\overline{E}}, \mathbb{W})$ is an unramified representation of W_E.[18] We obtain the following corollary.

Corollary 5.7.6. *If ℓ is coprime to the order of $\pi_0(Z(G))$, then there exist mutually commuting partial Frobenius operators σ_i acting on $R\Gamma(\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})_{\overline{E}}, \mathbb{W})$, such that for any lift σ of Frob_q, we have*

$$\sigma = \prod_{i=1}^r \sigma_i$$

as endomorphisms of $R\Gamma(\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})_{\overline{E}}, \mathbb{W})$. Moreover, each σ_i is annihilated by the renormalized Hecke polynomial $H_{\mu_i}(X) \in \mathcal{H}_{\mathcal{G}_i}[X]$ constructed in Section 5.6.5.

[18] This follows from the existence of smooth integral models due to Kisin [Kis10, Main theorem] and Kim–Madapusi [KMP16, Theorem 1], and the comparison between the cohomology of the special fiber and the generic fiber of Lan–Stroh [LS18, Corollary 4.6].

Proof. Let $\mathcal{V}_\mu \simeq \boxtimes_{i=1}^r \mathcal{V}_{\mu_i}$ be the decomposition as in the proof of Theorem 5.7.3. For each $\mathcal{V}_{\mu_i}$, we can apply the discussion in Section 5.6. Namely, for each element $\gamma \in W_{E_i}$, its action on $\mathcal{V}_{\mu_i}$ is annihilated by a (renormalized) minimal polynomial $H_{\mu_i,\gamma}(X)$, with coefficients in the spectral Bernstein center for H_i. Choose an arithmetic Frobenius element $\sigma \in W_E$ and denote its image under the map $W_E \to \prod_i W_{E_i} \to W_{E_i}$ by σ_i. We thus have the relation

$$\sigma = \boxtimes_{i=1}^r \sigma_i$$

as endomorphisms on $\mathcal{V}_\mu$. It is annihilated by the polynomial

$$H_{\mu,\sigma}(X) = \prod_{i=1}^r H_{\mu_i,\sigma_i}(X).$$

When restricted to the unramified locus on Par_G, i.e. where the universal L-parameter (up to $\widehat{G}$-conjugation) is trivial on the inertia subgroup of $W_{\mathbb{Q}_p}$, this polynomial, as well as each H_{μ_i,σ_i}, becomes independent of the choice of σ. The resulting polynomial we get is the Hecke polynomial and we relabel it as $H_\mu(X)$. It has coefficients in $\mathcal{H}_{\mathcal{G}}$; similarly we have the $H_{\mu_i}(X)$'s, which have coefficients in the $\mathcal{H}_{\mathcal{G}_i}$'s. We can now invoke the argument in Theorem 5.6.6 to conclude the desired result about the cohomology of Shimura varieties. $\square$

Remark 5.7.7. For $E = \mathbb{Q}_p$, Corollary 5.7.6 is also proved by Lee [Lee22, Theorem 1.0.4] by a completely different method.

6. Cohomology of some compact Shimura varieties

In this section we study the cohomology of compact Shimura varieties of abelian type. We focus on particular examples in types $A, B, C, D^{\mathbb{R}}$, and $D^{\mathbb{H}}$ in which the existence of global Galois representations or L-parameters is known, and where these are known to be attached to Hecke-eigensystems appearing in the cohomology of Shimura varieties. We then use Theorem 5.5.10 to prove a weak local-global compatibility for these Galois representations or L-parameters, see Corollary 6.1.2. For certain classical groups of type A, B, D, and C_2, we prove a strong local-global compatibility, see Corollary 6.4.2. This will be used in Section 7 to prove Theorem II.

6.1. **Abstract setup.** Let (G, X) be a Shimura datum of abelian type with reflex field E, and let v be a finite place of E above a rational prime p. Assume that G^{ad} is $\mathbb{Q}$-anisotropic, so that the Shimura varieties for (G, X) are projective. We let W be an algebraic representation of G^c over $\overline{\mathbb{Q}}_\ell$, for some $\ell \neq p$ with ℓ coprime to the order of $\pi_0(Z(G))$. We denote by $\mathbb{W}$ the associated ℓ-adic local system on $\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})$ for $K \subset \mathsf{G}(\mathbb{A}_f)$ neat, and write

$$H^i_{\mathrm{sm}}(\mathbf{Sh}(\mathsf{G}, \mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{W}) := \varinjlim_K H^i(\mathbf{Sh}_K(\mathsf{G}, \mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{W}),$$

equipped with its continuous and commuting actions of $\mathrm{Gal}_{\mathsf{E}}$ and $\mathsf{G}(\mathbb{A}_f)$.

6.1.1. Fix an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Recall that Matsushima's formula [BW00, Theorem 3.2] implies that we have a direct sum decomposition

$$H^*_{\mathrm{sm}}(\mathbf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{W})(d/2) = \bigoplus_{\Pi^\infty} \Pi^\infty \otimes_{\overline{\mathbb{Q}}_\ell} \sigma^*(\Pi^\infty),$$

of smooth $\mathsf{G}(\mathbb{A}_f) \times \mathrm{Gal}_{\mathsf{E}}$-representations, where $\mathsf{G}(\mathbb{A}_f)$ acts on Π^∞, and $\mathrm{Gal}_{\mathsf{E}}$ acts on $\sigma^*(\Pi^\infty) \coloneqq \mathrm{Hom}(\Pi^\infty, H^*(\mathbf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}, \mathbb{W})(d/2))$, which is the isotypic part for Π^∞ as a Hecke module. The direct sum runs over smooth representations Π^∞ of $\mathsf{G}(\mathbb{A}_f)$ over $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ such that there exists a $(\mathfrak{g}, K)$-module Π_∞ over $\mathbb{C}$ such that $\Pi = \Pi^\infty \otimes \Pi_\infty$ is a discrete automorphic representation of $\mathsf{G}(\mathbb{A})$, and such that Π_∞ is cohomological with infinitesimal character equal to that of W.

Corollary 6.1.2. *Assume that ℓ is coprime to the order of $\pi_0(Z(G))$. Fix Π^∞ as above, and let $\sigma \subset \sigma^i(\Pi^\infty)$ be a W_E-subrepresentation of dimension equal to $\dim V_\mu$. If σ is W_E-multiplicity free*[19]*, then $\sigma^{\mathrm{ss}} \simeq r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$ as representations of W_E.*

Proof. Let χ be the character of $\mathcal{Z}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ corresponding to Π_p and consider the localization $H^i_{\mathrm{sm}}(\mathbf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}\mathbb{W})_\chi$ of $H^i_{\mathrm{sm}}(\mathbf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}\mathbb{W})$ at χ. Then it follows from the semisimplicity of the Hecke action discussed above that we have a direct sum decomposition

$$H^i_{\mathrm{sm}}(\mathbf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}\mathbb{W})_\chi \simeq \bigoplus_{\chi_{\Pi_{1,p}} = \chi} \Pi_1^\infty \otimes_{\overline{\mathbb{Q}}_\ell} \sigma^i(\Pi_1^\infty),$$

where Π_1 runs over cuspidal automorphic representations of $\mathsf{G}(\mathbb{A})$ such that $\Pi_{1,\infty}$ is cohomological for W. A choice of nonzero vector v in Π^∞ thus exhibits

$$v \otimes \sigma \subset \Pi^\infty \otimes \sigma \subset H^i_{\mathrm{sm}}(\mathbf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\mathsf{E}}}\mathbb{W})_\chi$$

as a subrepresentation. It then follows from Theorem 5.5.10 that all the irreducible W_E-subquotients of σ also occur as irreducible W_E-subquotients of $r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$. Since σ has the same dimension as $r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$ and is multiplicity free, it follows that $r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$ is also multiplicity free. We deduce that σ and $r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$ have the same set of irreducible W_E-subquotients, and thus have isomorphic semisimplifications. To finish the proof, we note that $\phi^{\mathrm{FS}}_{\Pi_p}$ is semisimple and hence $r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$ is semisimple. Indeed, since we are in characteristic zero $\phi^{\mathrm{FS}}_{\Pi_p}$ being semisimple is equivalent to the statement that the identity component of the Zariski closure of the image of $\phi^{\mathrm{FS}}_{\Pi_p}$ is reductive, see e.g. [BMR05, Section 2.2]. This is clearly preserved under restriction to W_E and composition with r_μ, and thus $r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_p}\big|_{W_E}$ is semisimple. □

Remark 6.1.3. We think of Corollary 6.1.2 as a way to turn knowledge of the cohomology of Shimura varieties into knowledge about the Fargues–Scholze local Langlands correspondence; this is how it will be applied in the rest of this article.

[19]That is, its Jordan–Hölder factors are pairwise non-isomorphic.

6.2. **Examples.** We will now specialize the setting of the previous section to one of the following cases. In all cases $\mathsf{L}^+ \neq \mathbb{Q}$ will denote a totally real field.

A Let L be a quadratic CM extension of L^+ and let H be an inner form of the quasi-split unitary group (defined by an L/L^+-hermitian space) over L^+ such that for precisely one embedding $\tau_0 : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ is isomorphic to $\mathrm{U}(1, n-1)$ for $n > 2$, and for all other real embeddings τ the group has compact modulo center $\mathbb{R}$-points. We let $N(\mathsf{H}) = n$.

B Let V be a quadratic space over L^+ with special orthogonal group H, such that for precisely one embedding $\tau_0 : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ is isomorphic to $\mathrm{SO}(2, 2n-1)$, and for all other real embeddings τ the group is isomorphic to $\mathrm{SO}(0, 2n+1)$. In this case we set $\mathsf{L} = \mathsf{L}^+$ and $N(\mathsf{H}) = 2n$.

C Let H be an inner form of a split general symplectic group $\mathrm{GSp}_{2g,\mathsf{L}^+}$ such that for precisely one embedding $\tau_0 : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ is split, and for all other embeddings $\tau : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau} \mathbb{R}$ has compact modulo center $\mathbb{R}$-points. In this case we set $\mathsf{L} = \mathsf{L}^+$ and $N(\mathsf{H}) = 1 + \frac{g(g+1)}{2}$.

$D^{\mathbb{R}}$ Let V be a quadratic space over L^+ with special orthogonal group H, such that for precisely one embedding $\tau_0 : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ is isomorphic to $\mathrm{SO}(2, 2n-2)$, and for all other real embeddings τ the group is isomorphic to $\mathrm{SO}(0, 2n)$. In this case we set $\mathsf{L} = \mathsf{L}^+$ and $N(\mathsf{H}) = 2n$.

$D^{\mathbb{H}}$ Let n be an integer and let L be an extension of L^+ which is totally imaginary quadratic if n is odd and equal to L^+ if n is even. Let H^* be the (quasi-split) connected reductive group denoted by $\mathrm{GSO}_{2n}^{\mathsf{L}/\mathsf{L}^+}$ in [KS24, beginning of Section 6]. Let H be an inner form of H^* such that for precisely one embedding $\tau_0 : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ is isomorphic to the group GSO_{2n}^J of [KS24, Section 8, discussion before Lemma 8.1], and for all other embeddings $\tau : \mathsf{L}^+ \to \mathbb{R}$ the group $\mathsf{H} \otimes_{\mathsf{L}^+,\tau} \mathbb{R}$ has compact modulo center $\mathbb{R}$-points. We let $N(\mathsf{H}) = 1 + \frac{n(n-1)}{2}$.

In all of these cases we set $\mathsf{G} = \mathrm{Res}_{\mathsf{L}^+/\mathbb{Q}} \mathsf{H}$ and equip it with the Shimura datum X which is trivial at all real embeddings except τ_0, and which at τ_0 is given by:

$D^{\mathbb{H}}$ For $\epsilon \in \{-, +\}$, the Shimura datum X^{ϵ} described in [KS24, Equation (9.1)].

other cases The conjugacy class of the morphism $\mathbb{S} \to \mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ described in [Mil05, Appendix B]. To be precise, in type $B, D^{\mathbb{R}}$, the morphism described in loc. cit. lands in $\mathrm{GSpin}(V)$ and we consider its composition with $\mathrm{GSpin}(V) \to \mathrm{SO}(V)$.

It follows from the classification of abelian type Shimura data, see [Mil05, Appendix B], that (G, X) is of abelian type. It follows from a standard argument, see e.g. [Del79, Remark 2.3.12] that the reflex field is given by $\mathsf{L} \subset \mathbb{C}$, where the embedding $\mathsf{L} \to \mathbb{C}$ is one that extends $\tau_0 : \mathsf{L}^+ \to \mathbb{R}$. Note that since $\mathsf{L}^+ \neq \mathbb{Q}$, the group G^{ad} is $\mathbb{Q}$-anisotropic.

6.2.1. Fix a rational prime p and let $G = \mathsf{G} \otimes \mathbb{Q}_p$. Then we have a direct product decomposition

$$G = \prod_{\mathfrak{p}|p} \mathrm{Res}_{\mathsf{L}^+_{\mathfrak{p}}/\mathbb{Q}_p} \mathsf{H}_{\mathsf{L}^+_{\mathfrak{p}}}$$

over primes $\mathfrak{p}$ above p in L^+, and

$$G_{\overline{\mathbb{Q}}_p} = \prod_{\tau} \mathsf{H} \otimes_{\mathsf{L}^+,\tau} \overline{\mathbb{Q}}_p \tag{6.2.1}$$

over embeddings $\tau : \mathsf{L}^+ \to \overline{\mathbb{Q}}_p$. We have the following description of the dual group of $H_{\mathfrak{p}} = \mathsf{H}_{\mathsf{L}^+_{\mathfrak{p}}}$:

- A If $\mathfrak{p}$ splits in L, then $H_{\mathfrak{p}} = \mathrm{GL}_n$ and its dual group is GL_n with trivial $W_{\mathsf{L}^+_{\mathfrak{p}}}$-action. If $\mathfrak{p}$ does not split, then the dual group of $H_{\mathfrak{p}}$ is given by GL_n with $W_{\mathsf{L}^+_{\mathfrak{p}}}$-action factoring through $\mathrm{Gal}(\mathsf{L}_{\mathfrak{p}}/\mathsf{L}^+_{\mathfrak{p}})$ with the nontrivial element acting by transpose inverse.
- B The dual group of $H_{\mathfrak{p}}$ is given by Sp_{2n} with trivial $W_{\mathsf{L}^+_{\mathfrak{p}}}$-action.
- C The dual group of $H_{\mathfrak{p}}$ is given by the general spinor group GSpin_{2n+1} with trivial $W_{\mathsf{L}^+_{\mathfrak{p}}}$-action, see [KS23a, Section 1].
- $D^{\mathbb{R}}$ The dual group of $H_{\mathfrak{p}}$ is given by SO_{2n} with the action of $W_{\mathsf{L}^+_{\mathfrak{p}}}$ determined by the discriminant $\mathrm{Disc}(V \otimes_{\mathsf{L}^+} \mathsf{L}^+_{\mathfrak{p}}) \in (\mathsf{L}^+_{\mathfrak{p}})^\times/((\mathsf{L}^+_{\mathfrak{p}})^\times)^2$ of the quadratic space $V \otimes_{\mathsf{L}^+} \mathsf{L}^+_{\mathfrak{p}}$: If the discriminant is 1, then the action of $W_{\mathsf{L}^+_{\mathfrak{p}}}$ is trivial. Otherwise, the discriminant determines a quadratic extension $K/\mathsf{L}^+_{\mathfrak{p}}$ and the action of $W_{\mathsf{L}^+_{\mathfrak{p}}}$ factors through $\mathrm{Gal}(K/\mathsf{L}^+_{\mathfrak{p}})$ via the unique outer automorphism of SO_{2n}.
- $D^{\mathbb{H}}$ The dual group of $H_{\mathfrak{p}}$ is given by GSpin_{2n} with trivial $W_{\mathsf{L}^+_{\mathfrak{p}}}$-action if n is even, and if n is odd with $W_{\mathsf{L}^+_{\mathfrak{p}}}$-action factoring through $\mathrm{Gal}(\mathsf{L}_{\mathfrak{p}}/\mathsf{L}^+_{\mathfrak{p}})$ with the nontrivial element acting by the involution θ of [KS24, discussion before Lemma 2.3].

6.2.2. Fix an isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_p$, which induces a morphism $\tau_0 : \mathsf{L} \to \overline{\mathbb{Q}}_p$. The induced conjugacy class of cocharacters μ of $G(\overline{\mathbb{Q}}_p)$ induced by v and X is trivial (in the decomposition of (6.2.1)) in the factors corresponding to $\tau \neq \tau_0$. The cocharacter μ_{τ_0} corresponds to the following (highest weight) representation of the dual group of H:

- A The identity $\mathrm{GL}_n \to \mathrm{GL}_n$.
- B The standard representation $\mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$.
- C The spin representation $\mathrm{GSpin}_{2n+1} \to \mathrm{GL}_{2^n}$.
- $D^{\mathbb{R}}$ The standard representation $\mathrm{SO}_{2n} \to \mathrm{O}_{2n} \to \mathrm{GL}_{2n}$.
- $D^{\mathbb{H}}$ The half-spin representation $\mathrm{spin}^\epsilon : \mathrm{GSpin}_{2n} \to \mathrm{GL}_{2^{n-1}}$ of [KS24, Definition 2.6], see [KS24, Section 4, Equation (4.2)].

We denote by $\sim^0$ the equivalence relation on $\widehat{\mathsf{H}}$-valued L-parameters given by $\widehat{\mathsf{H}}$-conjugacy in types A, B, C, and $D^{\mathbb{H}}$, and given by O_{2n}-conjugacy in type $D^{\mathbb{R}}$.

6.3. **The existence of global Galois representations.** In this section we state the existence of global Galois representations or L-parameters for the groups in Section 6.2, under some assumptions.

6.3.1. Recall the global L-group ${}^{\mathsf{L}}\widehat{\mathsf{H}} = \widehat{\mathsf{H}} \rtimes \mathrm{Gal}_{\mathsf{L}^+}$, considered as a group scheme over $\mathbb{Q}$, where the Galois group is considered as a pro-finite algebraic group.

6.3.2. For a finite place v of L^+ we also consider the local L-group

$$ {}^{L}\widehat{\mathsf{H}_v} = \widehat{\mathsf{H}} \rtimes W_{\mathsf{L}_v^+}. $$

If H is unramified at v, then we recall from [BG14, Section 2.2] that to an irreducible smooth unramified representation π of $\mathsf{H}(\mathsf{L}_v^+)$ with $\mathbb{C}$-coefficients, we may associate an $\widehat{\mathsf{H}}(\mathbb{C})$-conjugacy class of morphisms $\phi_\pi : W_{\mathsf{L}_v^+} \to {}^{L}\widehat{\mathsf{H}_v}$; this is the Satake parameter of π.

6.3.3. Recall that for a finite place v' of L above v, the cocharacter μ (depending on v') induces a representation

$$ r_\mu : \widehat{\mathsf{H}} \rtimes W_{\mathsf{L}_{v'}} \to \mathrm{GL}(V_\mu). \tag{6.3.1} $$

Note that the action of $W_{\mathsf{L}_{v'}}$ on $\widehat{\mathsf{H}}$ is trivial in all cases, except in type $D^{\mathbb{R}}$ where it factors through the natural map $W_{\mathsf{L}_v^+} = W_{\mathsf{L}_{v'}} \to \mathrm{Gal}(K/\mathsf{L}_v^+)$. If $K \neq \mathsf{L}_v^+$, then the semidirect product $\widehat{\mathsf{H}} \rtimes \mathrm{Gal}(K/\mathsf{L}_v^+) = \mathrm{SO}_{2n} \rtimes \mathrm{Gal}(K/\mathsf{L}_v^+)$ is isomorphic to O_{2n}, see [GGP12, first Table in Section 7]. Moreover, the representation r_μ is the standard representation $\mathrm{O}_{2n} \to \mathrm{GL}_{2n}$; this is well known.[20]

6.3.4. To state the existence of global Galois representations, we need to formulate some assumptions. Let Π be a cuspidal automorphic representation of $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+})$, with H as in Section 6.2, and consider the following assumptions on Π and H. Let S_{sc} be the set of finite places w of L^+ where Π_w is supercuspidal.

- **coh** For all real places ∞ of L^+, the representation Π_∞ is cohomological in the sense of [KS23a, Definition 1.12].
- **coh-reg** Assumption **coh** holds. Moreover, the representation Π is std-regular in the sense of [Shi24, Definition 3.2.1].

[20] We recall that r_μ is the unique extension of the standard representation of SO_{2n}, such that $1 \rtimes \mathrm{Gal}(K/\mathsf{L}_v^+)$ acts trivially on the highest weight vector, see [Kot84, Lemma 2.1.2]. Consider the description of $\mathrm{SO}_{2n} \subset \mathrm{GL}_{2n}$ of [FH91, Section 18.1] with T the torus $(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$, and B the Borel given by the stabilizer of the isotropic flag $\langle e_1 \rangle \subset \cdots \subset \langle e_1, \dots, e_n \rangle$. Then the element $g \in \mathrm{GL}_{2n}$ swapping e_n and e_{2n} lies in O_{2n}, and conjugation by g induces the correct outer automorphism of SO_{2n}. It also preserves the highest weight vector, which is e_1.

ssc In cases A, B, and $D^{\mathbb{R}}$, there is a place w_{s} of L^+ such that the classical L-parameter[21] of $\Pi_{w_{\mathrm{s}}}$ has the property that $r_\mu \circ \phi_{\Pi_{w_{\mathrm{s}}}}$ is irreducible, cf. [Pen25b, Section 2.2].

simgen In cases A, B, and $D^{\mathbb{R}}$, the formal (Arthur) parameter Ψ of Π consists of a single cuspidal representation, see [Pen25b, Theorem 4.2.1, Definition 4.3.3 of v3].[22]

St There is a finite place v_{st} of L^+ such that $\Pi_{v_{\mathrm{st}}}$ is isomorphic to an unramified character twist of the Steinberg representation of $\mathsf{H}(\mathsf{L}^+_{v_{\mathrm{st}}})$. Furthermore, the group H is quasi-split at all finite places except possibly at $w \in S_{\mathrm{sc}}$ and v_{st}. In type C and $D^{\mathbb{H}}$, we further assume that H is quasi-split at $v \in S_{\mathrm{sc}}$.

St′ Assumption **St** holds. Moreover, in cases A, B, and $D^{\mathbb{R}}$, the place of $\mathbb{Q}$ underlying v_{st} is inert in L^+ and is not 2.

Then we have the following theorem due to many mathematicians.

Theorem 6.3.5. *Assume* [Shi24, Assumption H1]. *Fix any rational prime ℓ and an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. Let Π be a cuspidal automorphic representation of $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+})$.*

(*i*) *If H is of type C or $D^{\mathbb{H}}$ and Π satisfies* **coh-reg** *and* **St**, *then there is a unique (up to $\sim^0$) continuous semisimple Galois representation*

$$\rho_\Pi : \mathrm{Gal}_{\mathsf{L}^+} \to {}^{\mathsf{L}}\widehat{\mathsf{H}}(\overline{\mathbb{Q}}_\ell)$$

satisfying: a) Its composition with the projection ${}^{\mathsf{L}}\widehat{\mathsf{H}}(\overline{\mathbb{Q}}_\ell) \to \mathrm{Gal}_{\mathsf{L}^+}$ is the identity map. b) For all finite places $w \neq v_{\mathrm{st}}$, $w \nmid \ell$ of L^+ where H and Π are unramified, we have

$$\rho_\Pi|_{W_{\mathsf{L}^+_w}} \sim \iota_\ell(\phi_{\Pi_w}) \otimes \chi_\ell^{\frac{-1+N(\mathsf{H})}{2}}$$

where ϕ_{Π_w} is as in Section 6.3.2 and χ_ℓ is the ℓ-adic cyclotomic character.

(*ii*) *If H is of type A, B, or $D^{\mathbb{R}}$ and Π satisfies* **coh-reg**, **ssc**, **St**, **simgen**, *then there is a unique continuous semisimple Galois representation*

$$\rho_\Pi : \mathrm{Gal}_{\mathsf{L}} \to \mathrm{GL}(V_\mu)(\overline{\mathbb{Q}}_\ell)$$

such that for all finite places $w \nmid \ell$ of L with restriction w^+ to L^+, we have

$$\rho_\Pi|_{W_{\mathsf{L}_w}} \simeq \iota_\ell(r_\mu \circ \phi_{\Pi_{w^+}}|_{W_{\mathsf{L}_w}}) \otimes \chi_\ell^{\frac{-1+N(\mathsf{H})}{2}},$$

where $\phi_{\Pi_{w^+}}$ is the classical L-parameter of Π_{w^+}.

---

[21] By the *classical L-parameter* of Π_w we mean the L-parameter associated to Π_w by the local Langlands correspondences listed in Section 1.3.1, see [Mok15, Theorem 2.5.1, 3.2.1] and [KMSW14, Theorem 1.6.1] in type A, [Art13, Theorem 1.5.1] and [Ish24, Theorem 1.2] in type B, [Art13, Theorem 1.5.1] and [CZ21, Theorem A.2] in type $D^{\mathbb{R}}$, and [GT11a, Main Theorem] and [GT14, Main Theorem] in type C_2.

[22] The notation **simgen** stands for *simple generic*. We note that Π satisfies **simgen** precisely when its Arthur parameter is simple and generic in the sense of Arthur.

Proof. For part (i), we first transfer Π to a cuspidal automorphic representation $\Pi^\sharp$ of $\mathsf{H}^*(\mathbb{A}_{\mathsf{L}^+})$, where H^* is the quasi-split inner form of H, such that Π_w and $\Pi_w^\sharp$ are isomorphic at the unramified places w described in the theorem; this is possible by [KS23a, Proposition 6.3] and assumption **St**. Note that if Π is standard regular, then it is regular by [Shi24, Lemma 3.2.2]. Hence in particular, if it is ξ-cohomological, then ξ is regular. It now suffices to prove the analogous theorem for $\Pi^\sharp$. In type C, the result is [KS23a, Theorem A], and in type $D^{\mathbb{H}}$ it is [KS24, Theorem A].

Part (ii) follows from work of many authors, see [Pen25b, Theorem 4.3.6] for the statement and attributions, noting that our χ_ℓ is the inverse of his norm character $|-|$. Namely, first by [Art13], [Mok15], [KMSW14], [Ish24], [CZ25] we can reduce to the construction of Galois representations for cuspidal automorphic representations of GL_N and its local-global compatibility, which is due to [Clo90], [HT01], [TY07], [Shi11], [Car12], [CH13]. ☐

6.4. **Some local-global compatibility.** We now state some local-global compatibility for the Galois representations from the conclusion of Theorem 6.3.5.

6.4.1. In type C and $D^{\mathbb{H}}$ we define $R_\Pi = r_\mu \circ \rho_\Pi|_{W_{\mathsf{L}}}$. We have the following corollary of Theorem 6.3.5 and Corollary 6.1.2. **We will assume in the rest of Section 6 that** [Shi24, Assumption H1] **holds**.[23]

Corollary 6.4.2. *Let Π be a cuspidal automorphic representation of $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+})$ satisfying* **coh-reg**, **ssc**, **simgen** *and* **St$'$**. *Let ℓ be a prime together with an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, and assume that $\ell \neq 2$ in type $D^{\mathbb{R}}$. If v is a prime of L not dividing ℓ, with induced prime v^+ of L^+ such that $v^+ \neq v_{\mathrm{st}}$ and such that $R_\Pi|_{W_{\mathsf{L}_v}}$ is a multiplicity free representation of W_{L_v}, then*

$$\left(R_\Pi|_{W_{\mathsf{L}_v}}\right)^{\mathrm{ss}}(d/2) \simeq r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_{v^+}}\Big|_{W_{\mathsf{L}_v}}.$$

Proof. We consider $\sigma^i(\Pi^\infty)^{\mathrm{ss}}$ as in Corollary 6.1.2, with i the dimension of the Shimura variety and with W the automorphic local system corresponding to the algebraic representation with respect to which Π_∞ is cohomological. In type C, it follows from [KS23a, Corollary 8.7, discussion before Theorem 9.1, Corollary 12.4] that under **coh** and **St** we have an isomorphism[24]

$$\sigma^i(\Pi^\infty)^{\mathrm{ss}} \simeq R_\Pi(d/2).$$

The same result holds in type $D^{\mathbb{H}}$ by [KS24, Theorem 9.6, Proposition 14.2] under **coh-reg** and **St**. Here we used Chebotarev density and Brauer–Nesbitt theorems. The result now follows from Corollary 6.1.2.

[23] The results of [KS24] and [KS23a] used in the proof of Corollary 6.4.2 below, also rely on the work of Arthur [Art13]. By the work of [AGI$^+$24], they are thus (only) conditional on [Shi24, Assumption H1].

[24] Note that our definition of $\sigma^i(\Pi^\infty)$ includes a Tate twist by $d/2$ (in type C we have $d = \frac{g(g+1)}{2}$). The L-parameter ρ_π (denoted by ρ_π^C in [KS23a, Theorem 9.1]) is normalized so that $r_\mu \circ \rho_\Pi$ occurs in the middle degree cohomology of the Shimura variety, see [KS23a, Theorem 10.3 and the discussion in the first paragraph of its proof], which means that $(r_\mu \circ \rho_\Pi)(d/2)$ occurs in $\sigma^d(\Pi^\infty)$. A similar comment applies to our citation of [KS24] in case $D^{\mathbb{H}}$.

In types A, B, and $D^{\mathbb{R}}$, the representation R_Π is irreducible because $R_\Pi|_{\mathrm{Gal}_{\mathsf{L}_{w_s}}}$ is irreducible by assumption **ssc**. By [Pen25b, Corollary 4.5.7], under assumptions **ssc**, **coh-reg**, **simgen** and **St′**, the $\mathrm{Gal}_{\mathsf{L}}$-representation $\sigma^i(\Pi^\infty)$ contains a subrepresentation σ isomorphic to R_Π.[25] The result now follows from Corollary 6.1.2 and Theorem 6.3.5.(ii). □

Remark 6.4.3. Assume that H is of type C or $D^{\mathbb{H}}$. A particular consequence of Corollary 6.4.2 is that (under the assumptions of the corollary), the representation

$$\left(r_\mu \circ \rho_\Pi|_{W_{\mathsf{L}_v}}\right)^{\mathrm{ss}}$$

only depends on Π_{v+} and not on Π. This result seems to be new in general.

6.4.4. We have the following consequence of Corollary 6.4.2.

Corollary 6.4.5. *Assume that H is of type A, B, or $D^{\mathbb{R}}$, or that H is of type C with $g = 2$. Let Π be a cuspidal automorphic representation of $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+})$ satisfying* **coh-reg**, **ssc**, **simgen** *and* **St′**. *Let ℓ be a prime together with an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, and assume that $\ell \neq 2$ in type $D^{\mathbb{R}}$. If $v \nmid \ell$ is a prime of L with induced prime v^+ of L^+ such that $v^+ \neq v_{\mathrm{st}}$ and such that $r_\mu \circ \phi_{\Pi_{v^+}}\Big|_{W_{\mathsf{L}_v}}$ is a multiplicity-free representation of W_{L_v}, then*

$$\left(\phi_{\Pi_{v^+}}\right)^{\mathrm{ss}} \sim^0 \phi^{\mathrm{FS}}_{\Pi_{v^+}},$$

where $\phi_{\Pi_{v^+}}$ denotes the classical L-parameter of Π_{v^+}.

Proof. By Corollary 6.4.2, we have an isomorphism

$$(R_\Pi|_{W_{\mathsf{L}_v}})^{\mathrm{ss}}(d/2) \simeq r_\mu \circ \phi^{\mathrm{FS}}_{\Pi_{v^+}}\Big|_{W_{\mathsf{L}_v}}. \tag{6.4.1}$$

Now we observe that $d = N(\mathsf{H}) - 1$, and we deduce from Theorem 6.3.5 that (in type C_2, we use [Sor10, Main Theorem (a)] instead[26])

$$\begin{aligned}(R_\Pi|_{W_{\mathsf{L}_v}})^{\mathrm{ss}}(d/2) &= (r_\mu \circ \phi_{\Pi_{v^+}}\Big|_{W_{\mathsf{L}_v}})^{\mathrm{ss}} \\ &\simeq r_\mu \circ \phi^{\mathrm{ss}}_{\Pi_{v^+}}\Big|_{W_{\mathsf{L}_v}},\end{aligned}$$

and recall our descriptions of r_μ in all the different types of Section 6.2.2. By [GGP12, Theorem 8.1], it follows that in types A, B, and $D^{\mathbb{R}}$, equation (6.4.1) implies

$$\phi^{\mathrm{ss}}_{\Pi_{v^+}} \sim^0 \phi^{\mathrm{FS}}_{\Pi_{v^+}}.$$

[25] Note that in the notation of [Pen25b], we have $N(\mathsf{G}) - 1 = \mathrm{Dim}_{\mathbb{C}}\,\mathsf{X} = d$; this follows from inspection in all three cases, see [Mil05, Appendix B]. Unwinding the definitions in loc. cit. and using [Pen25b, Corollary 4.5.7], we see that $R_\Pi(d/2)$ occurs in $\sigma^i(\Pi^\infty)$.

[26] The Galois representation constructed by Sorensen in loc. cit. agrees with the one constructed by Kret–Shin, because of the uniqueness proved in [KS23a, Theorem A].

In type C (where $\mathsf{L} = \mathsf{L}^+$), we know that $\widehat{\mathsf{H}} = \mathrm{GSpin}_5 \simeq \mathrm{GSp}_4$ and under this isomorphism the representation r_μ corresponds to the standard embedding $\mathrm{GSp}_4 \subset \mathrm{GL}_4$. For the similitude character $\mathrm{Sim} : \mathrm{GSp}_4 \to \mathrm{GL}_1$, we know that

$$\left(\mathrm{Sim} \circ \rho_\Pi|_{W_{\mathsf{L}^+_{v^+}}}\right)^{\mathrm{ss}}$$

corresponds to the central character of Π_{v^+} by [KS23b, Theorem A]. The same is true for

$$\mathrm{Sim} \circ \phi^{\mathrm{FS}}_{\Pi_{v^+}}$$

by [FS24, Theorem I.9.6.(iii)]. By [GT11a, Lemma 6.1], it follows from this and (6.4.1) that

$$\phi_{\Pi_{v^+}} \sim \phi^{\mathrm{FS}}_{\Pi_{v^+}}.$$

$\square$

Remark 6.4.6. The proof of Corollary 6.4.5 does not work for GSp_{2g} for $g > 2$, because L-parameters valued in GSpin_{2n+1} are not determined by their composition with the spin representation; the same should hold in type $D^{\mathbb{H}}$ for $n > 4$. Perhaps in the latter case for $n = 4$, one might expect to prove that the parameters agree up to the outer S_3 action on L-parameters.

7. Compatibility of local Langlands correspondences

In this section we use the Igusa stacks for abelian type Shimura varieties to show compatibility between the construction of local Langlands correspondence for classical groups by [Art13, Ish24, Mok15, KMSW14], and the semisimple local Langlands correspondence constructed by Fargues–Scholze. Recently, Peng [Pen25b] has established similar results for groups over unramified extensions of $\mathbb{Q}_p$ with $p > 2$, based on the method of [Ham25] and [BMHN24]. Building on his work, we generalize this compatibility to the ramified case. Compared to the method in the literature, the major difference lies in that our main geometric result allows us to directly relate Fargues–Scholze parameters to automorphic representations, whereas the previous literature relies on p-adic uniformization of the basic locus of Shimura varieties, which is available only in the unramified case. This argument was suggested to us by David Hansen and Linus Hamann. We thank them heartily.

7.1. **Statement of the compatibility.** Fix a rational prime p. Let F be a finite extension of $\mathbb{Q}_p$ and let G over F be a connected reductive group. We denote by W_F the Weil group of F and write ${}^L G$ for the L-group of G. This is a semi-direct product $\widehat{G} \rtimes W_F$, where the Langlands dual group $\widehat{G}$ is viewed as a split reductive group over $\mathbb{Z}$ equipped with an algebraic action by W_F that factors through a finite quotient. We choose another prime $\ell \neq p$ and fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Note that the positive square root $\sqrt{p} \in \mathbb{R}$ defines a square root $\sqrt{p} \in \overline{\mathbb{Q}}_\ell$ using ι.

7.1.1. Recall that an L-parameter for G is a continuous homomorphism

$$W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(\mathbb{C}),$$

that is semisimple on the W_F factor, algebraic on the SL_2 factor and commutes with the projections to W_F on the source and target. We let $\Phi(G)$ be the set of $\widehat{G}(\mathbb{C})$-conjugacy classes of L-parameters for G. An L-parameter is *tempered* if its restriction to W_F has image that projects to a bounded subset of $\widehat{G}(\mathbb{C})$. We let $\Phi_{\mathrm{temp}}(G) \subset \Phi(G)$ be the subset of equivalence classes of tempered L-parameters.

7.1.2. Let $\Pi(G)$ be the set of isomorphism classes of $\mathbb{C}$-valued irreducible representations of $G(F)$, and $\Pi(G)_{\mathrm{temp}} \subset \Pi(G)$ be the subset of isomorphism classes of tempered representations. We will consider a collection of maps with finite fibers, labeled by Levi subgroups $M \subseteq G$ (including G itself)

$$\mathrm{LL}_M : \Pi_{\mathrm{temp}}(M) \to \Phi_{\mathrm{temp}}(M),\ \pi \mapsto \phi_\pi.$$

We will call such a collection of maps a *candidate local Langlands correspondence for* G. For a candidate local Langlands correspondence $\{\mathrm{LL}_M\}_{M\subseteq G}$, we write $\Pi_\phi(G)$ for the fiber $\mathrm{LL}_G^{-1}(\phi)$ and call it the (candidate) L-packet of a Langlands parameter ϕ.

7.1.3. Using the unique existence of the Langlands quotient and a similar classification of L-parameters, a candidate Langlands correspondence extends to a construction of L-parameters for general (not necessarily tempered) irreducible representations. To state this, recall from [Taï25, Section 4.3] that for a parabolic subgroup $P \subset G$, with Levi quotient M, we have an L-embedding $\iota_M : {}^L M \to {}^L G$, which is well-defined up to conjugation by $\widehat{G}$.

Lemma 7.1.4. *Let $\{\mathrm{LL}_M\}_{M\subseteq G}$ be a candidate local Langlands correspondence for G. Then it extends to a collection of well-defined maps*

$$\mathrm{LL}_M : \Pi(M) \to \Phi(M)$$

from the set of all irreducible smooth representations of $M(F)$ to the set of $\widehat{M}$-conjugacy classes of L-parameters for M.

Proof. This is [SZ18, Proposition 7.1]. We briefly recall the constructions for the readers' convenience. Following the notation in *loc. cit.*, the Langlands classification theorem, see [Sil78, Theorem 4.1] or [SZ18, Theorem 1.4], gives a bijection between $\Pi(M)$ and the set of triples (P, σ, ν), where $P \subseteq M$ is a standard parabolic subgroup, σ is a tempered representation of its Levi quotient L_P and $\nu \in X_F^*(L_P)_{\mathbb{R}}$ is a regular positive (with respect to P) character. The bijection takes (P, σ, ν) to the unique irreducible quotient $J(P, \sigma, \nu)$ of the normalized parabolic induction $i_P^M(\sigma \otimes \chi_\nu)$, for an unramified character χ_ν determined by ν.

On the other hand, [SZ18, Section 4.6] gives a similar classification of L-parameters: There is a bijection between $\Phi(M)$ and the set of triples $(P, {}^t\phi, \nu)$, where $P \subseteq M$ is a standard parabolic, ${}^t\phi$ is a tempered L-parameter for L_P and $\nu \in X_F^*(L_P)_{\mathbb{R}}$ is a regular positive character. The bijection takes $(P, {}^t\phi, \nu)$ to the composition of a twist ${}^t\phi_{z(\nu)}$ of ${}^t\phi$ by ν, with the L-embedding $\iota_{L_P} : {}^L L_P \to {}^L M$.

Hence for any $\pi \in \Pi(M)$ corresponding to (P, σ, ν), we can define $\mathrm{LL}_M(\pi)$ to be the L-parameter corresponding to $(P, \mathrm{LL}_{L_P}(\sigma), \nu)$. $\square$

7.1.5. Below given a candidate local Langlands correspondence $\{\mathrm{LL}_M\}_{M\subseteq G}$ (on tempered representations), if we write $\mathrm{LL}_M(\pi)$ for a non-tempered representation π, we mean $\mathrm{LL}_M(\pi)$, using the above construction of LL_M.

7.1.6. We say that an L-parameter is *semisimple* if its restriction to $\mathrm{SL}_2(\mathbb{C})$ is trivial. In this case, we also identify a semisimple L-parameter as a group homomorphism $W_F \to {}^LG(\mathbb{C})$ that recovers the original L-parameter as the composition

$$W_F \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{pr}_1} W_F \to {}^LG(\mathbb{C}).$$

For each L-parameter ϕ, we attach to it a semisimple L-parameter ϕ^{ss} by precomposing ϕ with the twisted embedding

$$W_F \to W_F \times \mathrm{SL}_2(\mathbb{C}), \quad \gamma \mapsto (\gamma, \mathrm{diag}(|\gamma|^{\frac{1}{2}}, |\gamma|^{-\frac{1}{2}})),$$

with $|\cdot|$ being the composition $W_F \to W_F^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_F^{-1}} F^\times \xrightarrow{|\cdot|_F} \mathbb{R}_+$.

Remark 7.1.7. Through the isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, we can turn an L-parameter ϕ as defined above into an ℓ-adic version of the L-parameter $\psi : W_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$ by first passing to Weil–Deligne representations via the Jacobson–Morozov Theorem, and then using [Del73, Section 8], see [Tat79, Theorem 4.2.1]. By construction, the semisimplification of ψ as a representation agrees with ϕ^{ss} under ι.

7.1.8. An L-parameter $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG(\mathbb{C})$ is called *discrete* if its centralizer S_ϕ contains $Z(\widehat{G})^{\mathrm{Gal}_F}$ as a finite index subgroup. An L-parameter $W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG(\mathbb{C})$ is called *supercuspidal* if it is discrete and semisimple. The following lemma is well known.

Lemma 7.1.9. *Let ϕ be an L-parameter, if ϕ^{ss} is supercuspidal, then ϕ is already supercuspidal.*

Proof. It suffices to show $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial, so that $\phi = \phi^{\mathrm{ss}}$ is supercuspidal. Let T be the diagonal torus of SL_2. Note that the image of $T(\mathbb{C})$ under ϕ lies in the centralizer of ϕ^{ss}, since

$$\phi(\gamma, \mathrm{diag}(|\gamma|^{\frac{1}{2}}, |\gamma|^{-\frac{1}{2}})) \cdot \phi(e, t) = \phi(e, t) \cdot \phi(\gamma, \mathrm{diag}(|\gamma|^{\frac{1}{2}}, |\gamma|^{-\frac{1}{2}}))$$

for all $\gamma \in W_F$ and $t \in T(\mathbb{C})$. But by assumption, ϕ^{ss} is discrete, so up to a finite group, the projection of $\phi(T)$ to $\widehat{G}$ is contained in the center. In particular, up to a finite group, $\phi(T)$ is central in $\phi(\mathrm{SL}_2)$. This is possible only if $\phi(\mathrm{SL}_2)$ is itself finite. But SL_2 is connected, so $\phi(\mathrm{SL}_2)$ has to be trivial as desired. $\square$

7.1.10. We denote by $\Phi^{\mathrm{ss}}(G)$ the set of semisimple L-parameters. Moreover, for a collection of maps $\{\mathrm{LL}_M\}_{M\subseteq G}$ labeled by Levi subgroups of G, we denote by $\{\mathrm{LL}_M^{\mathrm{ss}}\}_{M\subseteq G}$ the collection of maps

$$\mathrm{LL}_M^{\mathrm{ss}} \coloneqq (-)^{\mathrm{ss}} \circ \mathrm{LL}_M : \Pi_{\mathrm{temp}}(M) \to \Phi_{\mathrm{temp}}(M) \to \Phi^{\mathrm{ss}}(M),$$

and call it the semi-simplification of $\{\mathrm{LL}_M\}_{M\subseteq G}$.

7.1.11. Recall that we have fixed an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, which induces a $\sqrt{p} \in \overline{\mathbb{Q}}_\ell$. For each irreducible smooth representation, and in particular for any $\pi \in \Pi_{\mathrm{temp}}(G)$, Fargues–Scholze constructed a semisimple[27] ℓ-adic L-parameter $\phi^{\mathrm{FS}}_{\iota\pi} : W_F \to {}^L G(\overline{\mathbb{Q}}_\ell)$, using the fixed $\sqrt{p} \in \overline{\mathbb{Q}}_\ell$. Their construction works uniformly for all connected reductive groups, and is independent of ℓ and ι by [Sch25, Theorem 1.1]. We consider the collection of maps

$$\{\mathrm{LL}^{\mathrm{FS}}_M : \Pi_{\mathrm{temp}}(M) \to \Phi^{\mathrm{ss}}(M), \pi \mapsto \iota^{-1}\phi^{\mathrm{FS}}_{\iota\pi}\}_{M\subseteq G},$$

where M runs through Levi subgroups of G, and call it the *Fargues–Scholze (semisimple) local Langlands correspondence for G*. For each semisimple L-parameter ϕ for M, we denote the fiber $\mathrm{LL}^{\mathrm{FS},-1}_M(\phi)$ by $\Pi^{\mathrm{FS}}_\phi(M)$, and call it the Fargues–Scholze L-packet.

For a candidate local Langlands correspondence $\{\mathrm{LL}_M\}_{M\subseteq G}$, one can ask about the compatibility between $\{\mathrm{LL}^{\mathrm{ss}}_M\}_M$ and $\{\mathrm{LL}^{\mathrm{FS}}_M\}_M$. We say that the candidate local Langlands correspondence $\{\mathrm{LL}_M\}_M$ for G is *compatible with the Fargues–Scholze local Langlands*, if $\iota\phi^{\mathrm{ss}}_\pi = \phi^{\mathrm{FS}}_{\iota\pi}$ for all $\pi \in \Pi_{\mathrm{temp}}(M)$ and all Levi subgroups $M \subseteq G$.

7.2. **Characterization of the semisimple local Langlands correspondence.** If $\{\mathrm{LL}_M\}_{M\subseteq G}$ is a candidate local Langlands correspondence for G, then we call the collection $\{\mathrm{LL}^{\mathrm{ss}}_M\}_{M\subseteq G}$ a (candidate) semisimple local Langlands correspondence. We give in this subsection a characterization of the semisimple local Langlands correspondence in terms of Fargues–Scholze's construction. We let $F/\mathbb{Q}_p$ be a finite extension and G/F be a connected reductive group as before. We start with the case of quasi-split groups.

7.2.1. *Quasi-split groups.* Let G/F be quasi-split and let $\{\mathrm{LL}_M\}_{M\subseteq G}$ be a candidate local Langlands correspondence for G. Consider the following assumptions:

(1) For each M and each parabolic subgroup $P \subset M$ (with Levi quotient denoted by L_P), if $\pi \in \Pi(M)$ is a subquotient of the normalized parabolic induction $i^M_P\rho$ for some $\rho \in \Pi(L_P)$, then

$$\mathrm{LL}^{\mathrm{ss}}_M(\pi) = \iota_{L_P} \circ \mathrm{LL}^{\mathrm{ss}}_{L_P}(\rho),$$

where ι_{L_P} is the L-embedding ${}^L L_P \to {}^L M$ as explained in Section 7.1.3.

(2) For each M and each $\phi \in \Phi_{\mathrm{temp}}(M)$ the following holds: The parameter ϕ is a supercuspidal L-parameter if and only if all elements of $\Pi_\phi(M)$ are supercuspidal representations.

(3) For each M, and each supercuspidal L-parameter $\phi \in \Phi_{\mathrm{temp}}(M)$, the L-packet $\Pi_\phi(M)$ is contained in $\Pi^{\mathrm{FS}}_\phi(M)$.

(4a) For each M, and each $\phi \in \Phi_{\mathrm{temp}}(M)$, there is a stable virtual character

$$\Theta^1_\phi = \sum_{\pi\in\Pi_\phi(M)} a_\pi \Theta_\pi,$$

with $a_\pi \neq 0$ for all $\pi \in \Pi_\phi(M)$, where Θ_π is the Harish-Chandra character of π.

[27] In the sense of [FS24, VIII.3.1].

(4b) For each M, the stable virtual characters Θ^1_ϕ are atomically stable, i.e., if a linear combination $\Theta = \sum_{\pi \in \Pi_\phi(M)} b_\pi \Theta_\pi$ is stable then Θ is a multiple of Θ^1_ϕ.

We will often refer to Assumption (4) which is the conjunction of Assumptions (4a) and (4b).

Remark 7.2.2. Assumption (4) implies Assumption

($4'$) For each M, each $\phi \in \Phi_{\mathrm{temp}}(M)$, and $\pi, \pi' \in \Pi_\phi(M)$, we have $\phi^{\mathrm{FS}}_\pi = \phi^{\mathrm{FS}}_{\pi'}$.

This follows from work of Hansen, see [Han26, Corollary 1.2].

Remark 7.2.3 (Warning!)**.** In general the L-parameter $\mathrm{LL}_M(\pi)$ does not have to factor through a Levi subgroup even if its semi-simplification does, like in the case of the Steinberg representation for $\mathrm{GL}_2/\mathbb{Q}_p$.

Remark 7.2.4. Assumption (1) is an expected property of a local Langlands correspondence, see for example [Taï25, Conjecture 6.1(10)]. It implies the "only if" direction of Assumption (2). The "if" direction of Assumption (2) is not expected in general if G is not quasi-split. For example, when $G = D^\times$ is the non-quasi-split inner form of $\mathrm{GL}_2/\mathbb{Q}_p$, then the unique existence of local Langlands correspondence is known. The trivial representation has a discrete but non-supercuspidal L-parameter in this case. Since the L-packets are singletons, this gives an example of a supercuspidal L-packet with non-supercuspidal L-parameter. But when G is quasi-split, this is conjectured to always hold, see [DR09, Section 3.5(iv)]. Note also that when G is quasi-split, so are its Levi subgroups. Hence the Assumption (2) is expected to always hold.

Remark 7.2.5. Varma proves in [Var24, Proposition 2.5.2, Lemma 2.5.7.(ii)] that Assumption (4) is a consequence of [Var24, Hypothesis 2.5.1] (by setting $\mathcal{O}_M = \{\mathrm{id}_M\}$ for all M), where atomicity follows from the Θ^1_ϕ forming a basis of the space $\mathrm{SD}^{\mathrm{temp}}(G)$ of stable tempered virtual characters. In particular, it holds for many quasi-split classical groups as listed in [Var24, Proposition 5.1.2], including special orthogonal and unitary groups.

Remark 7.2.6. These assumptions are known for the usual local Langlands correspondence for $G = \mathrm{GL}_n$ of [HT01]. Indeed, (1) and (2) are well known, (3) is [FS24, Theorem IX.7.4], and (4a) and (4b) are essentially vacuous because the L-packets are singletons and all virtual characters are stable.

Proposition 7.2.7. *Let G/F be quasi-split. If a candidate local Langlands correspondence $\{\mathrm{LL}_M\}_{M \subseteq G}$ for G satisfies Assumptions (1), (2), (3), ($4'$), then it is compatible with the Fargues–Scholze local Langlands. In other words, the semisimple local Langlands correspondence $\{\mathrm{LL}^{\mathrm{ss}}_M\}_{M \subseteq G}$ is uniquely determined by these assumptions.*

Proof. Let $M \subseteq G$ be a Levi subgroup of G and $\pi \in \Pi_{\mathrm{temp}}(M)$ be a tempered irreducible representation. If $\phi_\pi := \mathrm{LL}_M(\pi)$ is a supercuspidal L-parameter, then it agrees with $\mathrm{LL}^{\mathrm{FS}}_M(\pi)$ by Assumption (3).

Therefore, we may assume ϕ_π is not supercuspidal. Then by Assumption (2), there is some representation $\pi' \in \Pi_{\phi_\pi}(M)$ that is a subquotient of $i_P^M\rho$ for some parabolic subgroup $P \subset M$ with Levi L_P and some $\rho \in \Pi(L_P)$. By Assumption (4′), we may replace π by π' and hence assume π itself is of this form.

Moreover, we have $\phi_\pi^{\mathrm{ss}} = \iota_{L_P} \circ \mathrm{LL}_{L_P}^{\mathrm{ss}}(\rho)$ by Assumption (1). Since the Fargues–Scholze local Langlands is compatible with parabolic inductions, see [FS24, Corollary IX.7.3][28], we also have

$$\phi_\pi^{\mathrm{FS}} = \iota_{L_P} \circ \mathrm{LL}_{L_P}^{\mathrm{FS}}(\rho).$$

This reduces us to showing $\mathrm{LL}_{L_P}^{\mathrm{ss}}(\rho) = \mathrm{LL}_{L_P}^{\mathrm{FS}}(\rho)$. But by picking L_P to be the minimal Levi through which ϕ_π^{ss} factors, we can assume that $\mathrm{LL}_{L_P}^{\mathrm{ss}}(\rho)$ is supercuspidal. Since L_P is itself a Levi subgroup of G, we may use Lemma 7.1.9 and Assumption (3) to conclude. $\square$

7.2.8. *Non-quasi-split groups.* We now consider the case of extended pure inner forms of quasi-split groups via the theory of endoscopic transfer. This does not encompass all connected reductive groups over F, but the most important cases of them, see the discussion in [Kal16, Section 4.5].

Let G/F be a reductive group that is realized as an extended pure inner form of a quasi-split group G^* via (ξ, b), where ξ is an isomorphism $G^*_{\overline{F}} \xrightarrow{\sim} G_{\overline{F}}$ and $b \in B(G^*)$ is an element such that $\xi^{-1}\sigma(\xi) = \mathrm{Ad}(b(\sigma))$ for all $\sigma \in \mathrm{Gal}_F$. In this case, the triple $\mathbf{1} := (G^*, 1, \mathrm{id})$ is an extended endoscopic triple for G, see [Kal16, Definition 2] for the definition. We fix a Whittaker datum $\mathfrak{w}$ for G^*. Following [Han26], see Footnote 5 in *loc. cit.*, cf. [Var24, Remark 3.2.2(i)], we may and do normalize the transfer factor $\Delta_{[\mathfrak{w},\mathbf{1},b]}$ to be the Kottwitz sign $e(G) \in \{\pm 1\}$, which is defined in [Kot83]. Recall also the notion of matching functions (depending on the transfer factor) from [Kal16, Definition 3, Theorem 4].

Assume now that we have a candidate local Langlands correspondence $\{\mathrm{LL}_M\}_{M \subseteq G}$ for G, as well as one $\{\mathrm{LL}_{M^*}\}_{M^* \subseteq G^*}$ for G^*. Consider the following assumption:

(5) Suppose that $\{\mathrm{LL}_{M^*}\}_{M^* \subseteq G^*}$ satisfies Assumption (4a). For each $\phi \in \Phi_{\mathrm{temp}}(G) = \Phi_{\mathrm{temp}}(G^*)$, and for all matching functions $f^* \in \mathcal{C}_c^\infty(G^*(F))$, $f \in \mathcal{C}_c^\infty(G(F))$, there is an equality

$$\Theta_\phi^{\mathbf{1}}(f^*) = \sum_{\pi \in \Pi_\phi(G)} c_\pi \Theta_\pi(f),$$

for some coefficients $c_\pi \in \mathbb{C}^\times$ (depending on $\mathfrak{w}$, but not on f, f^*), where $\Theta_\phi^{\mathbf{1}}$ is the stable virtual character supported on the L-packet $\Pi_\phi(G^*)$ as in Assumption (4a). The same is true when G is replaced by any Levi subgroup $M \subseteq G$.

[28] Note that there the parabolic induction is unnormalized, while the L-embedding ${}^L L_P \hookrightarrow {}^L M$ is twisted by a cyclotomic cocycle. Using normalized parabolic induction cancels out this cyclotomic twist, see [FS24, Beginning of Section IX.7.1], also [DHKM24a, p.3 (1), cf. Section 3.1]. Note, however, to get the description of the twisted embedding as stated in [FS24, Section IX.7.1], one needs to identify the Borel of the Satake group differently, see [CvdHS25, Remark 6.23].

We have the following analogous statement to Proposition 7.2.7.

Proposition 7.2.9. *If $\{\mathrm{LL}_{M^*}\}_{M^* \subseteq G^*}$ satisfies Assumptions (1), (2), (3), (4a) and (4′), and if Assumption (5) holds, then $\{\mathrm{LL}_M\}_{M \subseteq G}$ is compatible with the Fargues–Scholze local Langlands.*

Before we give the proof of this, let us briefly recall the construction of Fargues–Scholze L-parameters in terms of distributions, as well as results from [Han26]. We start by recalling the action of the Bernstein center on the space of distributions, following [Hai14, Section 3.1, 3.2], [Var24, Section 2.1], [Han26, Section 2.1].

7.2.10. Let H/F be a connected reductive group and let $C_c^\infty(H(F))$ be the convolution algebra of locally constant functions $H(F) \to \mathbb{C}$ that are compactly supported. The $\mathbb{C}$-linear dual of $C_c^\infty(H(F))$ is called the space of distributions. It contains a subspace $\mathrm{D}(H)$ spanned by the Harish-Chandra characters $\Theta_\pi : f \mapsto \mathrm{tr}(\pi|f)$ attached to irreducible representations π. Distributions also have a convolution product which is defined if one of the distributions is *essentially compact* (which means that the convolution by it preserves compactly supported smooth functions), see [Hai14, Lemma 3.1.1].

7.2.11. By a result of Bernstein–Deligne, the Bernstein center $\mathcal{Z}(H)$ can be identified with the algebra of essentially compact $H(F)$-invariant distributions, [Hai14, Section 3.2]. In particular, $\mathcal{Z}(H)$ acts on both $\mathrm{D}(H)$ and $C_c^\infty(H(F))$ by convolution. This action on $\mathrm{D}(H)$ preserves the subspace $\mathrm{D}^{\mathrm{temp}}(H)$ spanned by Harish-Chandra characters of tempered irreducible representations. The most important property for us is that if π is a smooth irreducible representation with associated distribution Θ_π, then for each $z \in \mathcal{Z}(H)$ we have

$$z * \Theta_\pi = z_\pi \Theta_\pi$$

for some scalar z_π. The association $z \mapsto z_\pi$ defines a character (ring homomorphism) $\chi_\pi : \mathcal{Z}(H) \to \mathbb{C}$, and $\mathcal{Z}(H)$ acts on π through χ_π.

7.2.12. In [FS24, IX.0.3], Fargues–Scholze constructed a morphism

$$\Psi_H : \mathcal{Z}^{\mathrm{spec}}(H) \to \mathcal{Z}(H),$$

from the spectral Bernstein center to the usual Bernstein center (base-changed to $\overline{\mathbb{Q}}_\ell$ via the chosen isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$). The Fargues–Scholze L-parameter ϕ_π^{FS} attached to an irreducible representation π is constructed via the composition $\chi_\pi \circ \Psi_H$. Indeed, this composition corresponds to a point on the coarse moduli of the stack of L-parameters, and consequently a semisimple L-parameter by [FS24, Proposition VIII.3.2].

One can alternatively phrase the construction of Fargues–Scholze L-parameters in terms of distributions. Namely, through the map Ψ_H, there is an action of the spectral Bernstein center on the space of distributions $D(H)$. Then for any smooth irreducible representation π, and any $z \in \mathcal{Z}^{\mathrm{spec}}(H)$, $\Psi_H(z) * \Theta_\pi = z_\pi \Theta_\pi$, for some scalar $z_\pi \in \mathbb{C}$. In this way, we get a character $z \mapsto z_\pi$ of $\mathcal{Z}^{\mathrm{spec}}(H)$, and hence a semisimple L-parameter ϕ_π^{FS}.

7.2.13. According to [Han26, Theorem 1.1, 1.4], the image $\mathcal{Z}^{\mathrm{FS}}(H)$ of Fargues–Scholze's map Ψ_H lies in the subalgebra of very stable[29] central distributions. Moreover, if H is an extended pure inner form of a quasi-split group H^*, then the map Ψ_H factors through a surjective map $\tau_H : \mathcal{Z}^{\mathrm{FS}}(H^*) \to \mathcal{Z}^{\mathrm{FS}}(H)$. This is compatible with the usual transfer of tempered stable distributions in the following sense: Suppose Θ^*, resp. Θ are stable virtual characters of H^*, resp. H, supported on tempered irreducible representations. If for all matching functions $f^* \in \mathcal{C}_c^\infty(H^*(F))$, $f \in \mathcal{C}_c^\infty(H(F))$, there is an equality

$$\Theta^*(f^*) = \Theta(f),$$

then for any $z \in \mathcal{Z}^{\mathrm{spec}}(H) \simeq \mathcal{Z}^{\mathrm{spec}}(H^*)$, one has

$$(\Psi_{H^*}(z) * \Theta^*)(f^*) = (\Psi_H(z) * \Theta)(f).$$

We can now give the proof of Proposition 7.2.9.

Proof of Proposition 7.2.9. Let $M \subseteq G$ be a Levi subgroup. Let $\pi \in \Pi_{\mathrm{temp}}(M)$ be an irreducible tempered smooth representation and $\phi := \mathrm{LL}_M(\pi)$ be its attached L-parameter under LL_M.

By Assumption (4a), there exists a stable virtual character Θ_ϕ^1 supported on $\Pi_\phi(M^*)$. By Assumption (4′), for any $z \in \mathcal{Z}^{\mathrm{spec}}(M^*)$, the actions of $\Psi_{M^*}(z)$ on all members of $\Pi_\phi(M^*)$ will be through the same scalar. Hence for any such z, we have

$$(\Psi_{M^*}(z) * \Theta_\phi^{\mathbf{1}}) = z_\phi \Theta_\phi^{\mathbf{1}},$$

for some scalar $z_\phi \in \mathbb{C}$ that only depends on ϕ, but not the individual members of the L-packet. The association $z \mapsto z_\phi$ defines a nonzero character of $\mathcal{Z}^{\mathrm{spec}}(M^*)$, which corresponds to the Fargues–Scholze parameter of any member $\rho \in \Pi_\phi(M^*)$. We first want to compare this character with the one corresponding to ϕ_π^{FS}.

For this, we use Assumption (5) to transfer Θ_ϕ^1 to a virtual character

$$\sum_{\pi \in \Pi_\phi(M)} c_\pi \Theta_\pi,$$

which is supported on the candidate L-packet $\Pi_\phi(M) := \mathrm{LL}_M^{-1}(\phi)$. This is a stable (and in fact tempered) virtual character according to [Var24, Remark 3.2.2(ii)], which says that under transfer, stable distributions go to stable distributions.

But by [Han26, Theorem 1.4.ii] (see the explanation in Section 7.2.13), for any $z \in \mathcal{Z}^{\mathrm{spec}}(M) \simeq \mathcal{Z}^{\mathrm{spec}}(M^*)$, and all matching functions $f^* \in \mathcal{C}_c^\infty(M^*(F))$, $f \in \mathcal{C}_c^\infty(M(F))$, we have

$$\begin{aligned} z_\phi \left(\sum_{\pi \in \Pi_\phi(M)} c_\pi \Theta_\pi \right)(f) &= z_\phi \Theta_\phi^{\mathbf{1}}(f^*) = (\Psi_{M^*}(z) * \Theta_\phi^{\mathbf{1}})(f^*) \\ &= \sum_{\pi \in \Pi_\phi(M)} c_\pi (\Psi_M(z) * \Theta_\pi)(f) = \sum_{\pi \in \Pi_\phi(M)} c_\pi \cdot z_\pi \Theta_\pi(f), \end{aligned}$$

[29] That is, a stable distribution whose convolution with unstable functions gives unstable functions, see [Han26, Section 2.1, Lemma 2.2].

where $z_\pi \in \mathbb{C}$ is the scalar, through which $\mathcal{Z}^{\mathrm{spec}}(M)$ acts on π. If we compare the coefficients on both sides, then we see, by linear independence of the Harish-Chandra characters, that $z_\pi = z_\phi$ for all $\pi \in \Pi_\phi(M)$. Now as explained in Section 7.2.12, this means the Fargues–Scholze parameters for all members of the L-packet $\Pi_\phi(M)$ are the same, and they equal ϕ_ρ^{FS}, for any $\rho \in \Pi_\phi(M^*)$.

Now under our assumptions, we know the compatibility of $\{\mathrm{LL}_{M^*}\}_{M^* \subset G^*}$ with the Fargues–Scholze local Langlands correspondence by Proposition 7.2.7. Therefore, we have $\phi_\pi^{\mathrm{FS}} = \phi_\rho^{\mathrm{FS}} = \phi^{\mathrm{ss}}$ as desired. □

Remark 7.2.14. Assumption (5) is a form of an endoscopic character identity, which is expected as part of the refined local Langlands conjecture for non-quasi-split groups, see [Kal16, Conjecture F]. Note that here we only require the property that the L-packet for the parameter ϕ is the support of the endoscopic transfer of Θ_ϕ^1, whereas the refined conjecture in *loc. cit.* says that if one renormalizes the transfer factor $\Delta_{[\mathfrak{w},\mathbf{1},b]}$ suitably (which will only change it by a scalar and hence does not change the support of the transfer of Θ_ϕ^1), then one can precisely predict the coefficients c_π. These coefficients capture the inner structure of the L-packet.

7.3. **Compatibility for unitary and odd special orthogonal groups.** In this and the next subsection, we prove the compatibility between Fargues–Scholze local Langlands correspondence with previous constructions of local Langlands correspondences for extended pure inner forms of quasi-split unitary groups, odd special orthogonal groups, and GSp_4. Because the previous correspondences are constructed using the work of Arthur [Art13], **we will assume for the rest of Section 7 that** [Shi24, Hypothesis H1] **holds**. Therefore, we may appeal to [AGI+24] and unconditionally use the results of Arthur [Art13] and Mok [Mok15].

7.3.1. Let $F/\mathbb{Q}_p$ be a finite extension and G/F be a reductive group as before. We assume G is an extended pure inner form of a quasi-split group G^*, which is either the quasi-split unitary group attached to an Hermitian space with respect to a quadratic extension F'/F (type A), the split special orthogonal group $\mathrm{SO}_{2n+1,F}$ for some integer n (type B), or $\mathrm{GSp}_{4,F}$ (type C_2). We set $F' = F$ in cases B and C_2. We let $\{\mathrm{LL}_M\}_{M \subseteq G}$ be a (candidate) local Langlands correspondence for G in the list below.[30]

- Type A:
  (1) For $G = G^*$, the construction by Mok [Mok15, Theorem 2.5.1, 3.2.1],

[30] Here we are using the fact that Levi subgroups of unitary groups (resp. special orthogonal groups of odd rank) are isomorphic to products of general linear groups with unitary groups (resp. special orthogonal groups of odd rank), and in type C_2 that the Levi subgroups are products of inner forms of general linear groups. This allows us to define a local Langlands correspondence for these Levi subgroups M. Proving properties (1), (2), (3) and (4) for all Levi subgroups M follows then, by known properties of the local Langlands correspondence for GL_n, see Remark 7.2.6, by proving them for all quasi-split unitary (resp. odd orthogonal) groups; in type C_2 it is enough to deal with $M = G$.

   (2) For general G, by Kaletha–Minguez–Shin–White [KMSW14, Theorem 1.6.1].[31]
- Type B:
   (1) For $G = G^*$, by Arthur [Art13, Theorem 1.5.1],
   (2) For general G, by Ishimoto [Ish24, Theorem 1.2].
- Type C_2:
   (1) For $G = G^* = \mathrm{GSp}_{4,F}$ by [GT11a, Main theorem].
   (2) For general G, by [GT14, Main theorem].

Proposition 7.3.2. *Let F, G, G^* be as in Section 7.3.1. If G^* is of type A or B, then the local Langlands correspondences $\{\mathrm{LL}_{M^*}\}_{M^* \subseteq G^*}$ from the above list satisfy Assumptions (1), (2) and (4). If G^* is of type C_2, then the local Langlands correspondences $\{\mathrm{LL}_{M^*}\}_{M^* \subseteq G^*}$ satisfy Assumptions (1), (2), (4a) and (4′). For all non quasi-split G as above, the local Langlands correspondences $\{\mathrm{LL}_M\}_{M \subseteq G}$ from the above list satisfy Assumption (5).*

Proof. We first discuss type A and B: For Assumption (1), this is [Pen25b, Proposition 2.4.3], cf. [DHKM24b, Proposition 7.9]. That Assumption (2) is satisfied is due to Moeglin [Mœg07, Theorem 8.4.4], Moeglin [Mœg11, Theorem 1.5.1], cf. [Xu17, Theorem 3.3], see also [Pen25b, Corollary 2.5.2]. For Assumption (4), this is [Var24, Proposition 5.1.2], see Remark 7.2.5. When G is non-quasi-split, Assumption (5) is satisfied by [KMSW14, Theorem 1.6.1.4], [Ish24, Theorem 3.11.3]. In type C_2, Assumption (1) can be verified directly from the explicit description of the L-parameters for non-supercuspidal representations, see [GT11b, Proposition 13.1]. Assumption (2) is proved in [Ham25, discussion after Remark 2.2]. Assumption (4a) is [CG15, Main theorem] and Assumption (4′) follows from [Ham25, Proposition 8.1.(ii)] for non supercuspidal ϕ; for supercuspidal ϕ Assumption (4′) is vacuous since then the L-packet $\Pi_\phi(G^*)$ has cardinality one. Assumption (5) is [CG15, Proposition 11.1.(i)]. □

Remark 7.3.3. The reader might have noticed that our treatment of GSp_4 alongside the odd orthogonal and unitary cases is slightly artificial. For instance, for G of type C_2, the compatibility between LL_G and $\mathrm{LL}_G^{\mathrm{FS}}$ is known for those π with $\mathrm{LL}_G(\pi)$ not supercuspidal, by [Ham25, Proposition 8.1.(ii)]. Nevertheless, for the remaining π our argument is the same as in type A and B, and thus we have opted for the current presentation.

Before we prove the desired compatibility, we need some preparations. The first is a lemma about globalization. This is essentially [Art13, Lemma 6.2.1, 6.2.2], and [Mok15, Lemma 7.2.1, 7.2.3], except that we need to globalize the group in a way that it gives rise to a compact Shimura variety.

Lemma 7.3.4. *Let F, G be as before. Let $\pi \in \Pi_{\mathrm{temp}}(G)$ be a supercuspidal representation. Then there exists a tuple $(\mathsf{L}, \mathsf{L}^+, v, \mathsf{H}, \mathsf{X}, \Pi)$, where*

[31] We note that [KMSW14, Theorem 1.6.1] is stated for all representations (or L-parameters), but is only proved for tempered representations (or L-parameters) in [KMSW14, Theorem 1.6.1]. This does not affect us, since we only use their results for tempered representations.

- $\mathsf{L}^+ \neq \mathbb{Q}$ *is a totally real field,*
- v *is a* p*-adic place of* L^+*,*
- H *is a connected reductive group over* L^+*,*
- X *is a Shimura datum for* $\mathsf{G} = \mathrm{Res}_{\mathsf{L}^+/\mathbb{Q}} \mathsf{H}$ *with reflex field* L*, an extension of* L^+ *of degree* ≤ 2*,*
- Π *is a cuspidal automorphic representation of* $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+})$*;*

such that $\mathsf{L}^+_v \simeq F$*,* $\mathsf{H}_v \simeq G$*, such that* (G, X) *gives one of the Shimura data of Section 6.2, and* $\Pi_v \simeq \pi$*, satisfies the conditions* **coh-reg**, **ssc**, **simgen** *and* **St′**.

Proof. We first focus on types A and B. By [Art13, Lemma 6.2.1], [Mok15, Lemma 7.2.1], we can find a totally real number field L^+ (and a CM extension L/L^+ in the unitary case) with $\mathsf{L}^+_v \simeq F$ (and $\mathsf{L}_{v'} \simeq F'$ for a unique place v' above v) for a p-adic place v of L^+. Let H^* be the split odd special orthogonal group SO_{2n+1} or a quasi-split unitary group $\mathrm{U}_{\mathsf{L}/\mathsf{L}^+}(n)$ attached to L over L^+. Consider the short exact sequence

$$H^1(\mathsf{L}^+, \mathsf{H}^{*,\mathrm{ad}}) \to \bigoplus_w H^1(\mathsf{L}^+_w, \mathsf{H}^{*,\mathrm{ad}}_w) \to \pi_1(\mathsf{H}^{*\mathrm{ad}})_{\mathrm{Gal}_{\mathsf{L}^+}} \to 1,$$

which comes from [Kot86, Corollary 2.5, Proposition 2.6]. It follows from this that there is an inner form H of H^* such that:

- There is exactly one infinite place τ_0 of L^+ such that $\mathsf{H} \otimes_{\mathsf{L}^+,\tau_0} \mathbb{R}$ has non-compact $\mathbb{R}$-points. In the case of type B, H is isomorphic to $\mathrm{SO}(2, 2n-1)$ at τ_0, and in the case of type A, H is isomorphic to $\mathrm{U}(1, n-1)$ at τ_0;
- At the distinguished p-adic place v the group $\mathsf{H} \otimes_{\mathsf{L}^+} \mathsf{L}^+_v$ is isomorphic to G.

Let $\mathsf{G} = \mathrm{Res}_{\mathsf{L}^+/\mathbb{Q}} \mathsf{H}$, which is clearly $\mathbb{Q}$-anisotropic modulo center. It is equipped with a Shimura datum X that is nontrivial precisely at the factor of $\mathsf{H}_{\mathbb{R}}$ corresponding to τ_0, such that (G, X) is a Shimura datum. By construction, it corresponds to the setup in type A or B of Section 6.2.

To globalize the representation, we apply [Shi12, Theorem 4.8]. More precisely, we apply [KMSW14, Lemma 4.2.1, Lemma 4.3.1] in type A and [Ish24, Lemma 6.4, Lemma 6.7] in type B to construct a cuspidal automorphic representation Π^* of $\mathsf{H}^*(\mathbb{A}_{\mathsf{L}^+})$ satisfying **coh-reg**, **ssc**, **St′**, **simgen** and $\Pi_v \simeq \pi$. Here we use the fact that for an auxiliary place $w \nmid 2$ of L^+ where H_w is quasi-split and unramified, the group H_w admits supercuspidal representations satisfying **ssc**, see [Oi19b, Theorem 1.1] in type A and [Oi19a, Theorem 1.1] in type B. We are moreover using the fact that simple generic as in [KMSW14, Lemma 4.3.1] and [Ish24, Lemma 6.7] is the same as **simgen**. We then transfer Π^* to a cuspidal automorphic representation Π of $\mathsf{H}(\mathbb{A}_{\mathsf{L}^+})$ satisfying **coh-reg**, **ssc**, **St′**, **simgen**, using [Pen25b, Theorem 4.3.5].

In type C_2, we can let H be as in type C of Section 6.2 with $g = 2$, assuming that H is quasi-split at all finite places except possibly for one place called v_{st} (see [KS23a, Section 7] for the existence of such H). The globalization can be done using a routine application of [Shi12, Theorem 4.8]. $\square$

7.3.5. Let $\mathrm{Std} : \widehat{G} \to \mathrm{GL}(V)$ be the standard representation of the dual group, which we take to be $\mathrm{GSpin}_5 \simeq \mathrm{GSp}_4 \to \mathrm{GL}_4$ in type C_2. The following lemma is well known.

Lemma 7.3.6. *Let ϕ be a discrete and semisimple L-parameter. Then $\mathrm{Std} \circ \phi|_{W_{F'}}$ is multiplicity-free as a representation of $W_{F'}$.*

Proof. In type C_2, this is [GT11a, Lemma 6.2.(ii)]. In type A and B, this is a direct consequence of the descriptions of S_ϕ in [GGP12, Section 4] as we will now explain: Write $M = \mathrm{Std} \circ \phi|_{W_{F'}}$, which is conjugate self dual in type A, and self dual in type B. Following the notation of [GGP12, Section 4], we write

$$M = \bigoplus M_i^{\oplus m_i} \oplus \bigoplus_I N_i^{\oplus 2n_i} \oplus \bigoplus (P_i \oplus (P_i^\sigma)^\vee)^{\oplus p_i},$$

where each M_i, N_i, P_i is irreducible and the M_i, N_i, P_i are pairwise distinct. The group S_ϕ is then described in [GGP12, Section 4] (there it is denoted C) as

$$\prod_i \mathrm{O}(m_i) \times \prod_i \mathrm{Sp}(2n_i) \times \prod_i \mathrm{GL}(p_i).$$

Since ϕ is discrete, it follows that S_ϕ is finite and thus that all the p_i and n_i are zero, and that all the m_i are equal to one; the lemma follows. ☐

Remark 7.3.7. If π is supercuspidal, then $\phi = \mathrm{LL}_G(\pi)$ is a discrete L-parameter. However, if ϕ is not supercuspidal, then it is possible that ϕ^{ss} is not discrete and thus that $\mathrm{Std} \circ \phi\big|_{W_{F'}}$ is not multiplicity-free. This shows that we cannot directly prove Assumption (3) for such π using a globalization argument and Corollary 6.4.5, as happens for supercuspidal ϕ in the proof of Theorem 7.3.8 below.

Theorem 7.3.8. *Let F, G, G^* be as in Section 7.3.1. The local Langlands correspondences $\{\mathrm{LL}_M\}_{M \subseteq G}$ from the above list are compatible with the Fargues–Scholze local Langlands correspondence.*

Proof. By Propositions 7.2.7, 7.2.9 and 7.3.2, it suffices to check Assumption (3), namely the compatibility between the constructions on supercuspidal L-packets for the quasi-split inner form G^* (which is attached to some quadratic extension E/F in the unitary case). Hence we may and do assume $G = G^*$. Let $\phi \in \Phi_{\mathrm{temp}}(G)$ be a supercuspidal L-parameter and let $\pi \in \Pi_\phi(G)$ be a member of its L-packet. By Assumption (2), it is a supercuspidal representation.

We globalize the triple (F, G, π) using Lemma 7.3.4. In particular, we get a tuple $(\mathsf{L}, \mathsf{L}^+, v, \mathsf{G}, \mathsf{X}, \Pi)$ as in the conclusion of that lemma ($\mathsf{L} = \mathsf{L}^+$ when G is a special orthogonal group). The compatibility is then Corollary 6.4.5, using the multiplicity-freeness of Lemma 7.3.6.

Now note that any proper Levi subgroup $M \subset G$ is a product of groups from the same list provided at the beginning of this subsection, or inner forms of general linear groups, which are of strictly smaller ranks. Hence we may conclude the compatibility of $\{\mathrm{LL}_M\}_{M \subseteq G}$ with the Fargues–Scholze local Langlands correspondence as desired. ☐

7.4. **Even special orthogonal groups.** Let F be a p-adic local field as before. Let V' be a $2n$-dimensional quadratic space over F with quasi-split orthogonal group G^*, and let G be a pure inner form of G^*. Note that $G \simeq \mathrm{SO}(V)$ for some quadratic space V over F, and that the action of Gal_F on the absolute root datum of G is of order 1 or 2.[32] Write $G = G_b$ as an extended pure inner form of G^*. Note that $\widehat{G} = \mathrm{SO}_{2n}$, so that O_{2n} acts by conjugation on $\widehat{G}$. We define

$$\widetilde{\Phi}_{\mathrm{temp}}(G) = \Phi_{\mathrm{temp}}(G)/\,\mathrm{O}_{2n}(\mathbb{C}),$$

and similarly define $\widetilde{\Phi}^{\mathrm{ss}}(G)$ and consider $(-)^{\mathrm{ss}} : \widetilde{\Phi}_{\mathrm{temp}}(G) \to \widetilde{\Phi}^{\mathrm{ss}}(G)$. We consider the map

$$\widetilde{\mathrm{LL}}_G : \Pi_{\mathrm{temp}}(G) \to \widetilde{\Phi}_{\mathrm{temp}}(G)$$

constructed by Arthur [Art13, Theorem 1.5.1] in the quasi-split case and by Chen–Zou [CZ21, Theorem A.2] in the non quasi-split case. We also consider $\widetilde{\mathrm{LL}}_G^{\mathrm{FS}}$ constructed from $\mathrm{LL}_G^{\mathrm{FS}}$ by composing with $\Phi^{\mathrm{ss}}(G) \to \widetilde{\Phi}^{\mathrm{ss}}(G)$. This way we get L-packets $\widetilde{\Pi}_\phi(G)$ and $\widetilde{\Pi}_\phi^{\mathrm{FS}}(G)$.

Theorem 7.4.1. *We have the equality* $(-)^{\mathrm{ss}} \circ \widetilde{\mathrm{LL}}_G = \widetilde{\mathrm{LL}}_G^{\mathrm{FS}}$.

7.4.2. Before we prove the theorem, it is useful to reinterpret the set $\widetilde{\Phi}^{\mathrm{ss}}_{\mathrm{temp}}(G)$ geometrically. Recall that $Z^1(W_F, \mathrm{SO}_{2n})$ is the moduli space of SO_{2n}-valued L-parameters of W_F, see [FS24, Theorem VIII.1.3]. It is equipped with a natural conjugation action by O_{2n}, and we consider the diagram

$$\begin{array}{ccc} [Z^1(W_F, \mathrm{SO}_{2n})/\,\mathrm{SO}_{2n}] & \longrightarrow & Z^1(W_F, \mathrm{SO}_{2n}) /\!\!/ \mathrm{SO}_{2n} \\ \downarrow & & \downarrow \\ [Z^1(W_F, \mathrm{SO}_{2n})/\,\mathrm{O}_{2n}] & \longrightarrow & Z^1(W_F, \mathrm{SO}_{2n}) /\!\!/ \mathrm{O}_{2n}\,. \end{array}$$

It follows from [FS24, Proposition VIII.3.2] that SO_{2n}-conjugacy classes of semisimple L-parameters $W_F \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ are in bijection with closed SO_{2n}-orbits in $Z^1(W_F, \mathrm{SO}_{2n})(\overline{\mathbb{Q}}_\ell)$ and also with closed points in $(Z^1(W_F, \mathrm{SO}_{2n}) /\!\!/ \mathrm{SO}_{2n})(\overline{\mathbb{Q}}_\ell)$. The proof of that theorem also establishes that O_{2n}-conjugacy classes of semisimple L-parameters $W_F \to \mathrm{SO}_{2n}(\overline{\mathbb{Q}}_\ell)$ are in bijection with closed O_{2n}-orbits in $Z^1(W_F, \mathrm{SO}_{2n})(\overline{\mathbb{Q}}_\ell)$ and also with closed points in $(Z^1(W_F, \mathrm{SO}_{2n}) /\!\!/ \mathrm{O}_{2n})(\overline{\mathbb{Q}}_\ell)$.

Recall that $\mathcal{Z}^{\mathrm{spec}}(\mathrm{SO}_{2n})$ is the ring of functions of $Z^1(W_F, \mathrm{SO}_{2n}) /\!\!/ \mathrm{SO}_{2n}$ and we define $\widetilde{\mathcal{Z}}^{\mathrm{spec}}(\mathrm{SO}_{2n}) \subset \mathcal{Z}^{\mathrm{spec}}(\mathrm{SO}_{2n})$ to be the ring of functions of $Z^1(W_F, \mathrm{SO}_{2n}) /\!\!/ \mathrm{O}_{2n}$. Note that $\widetilde{\mathcal{Z}}^{\mathrm{spec}}(\mathrm{SO}_{2n}) \subset \mathcal{Z}^{\mathrm{spec}}(\mathrm{SO}_{2n})$ is just the subring of functions fixed by the involution induced by the O_{2n}-action.

Proof of Theorem 7.4.1. We first deal with the case that G is quasi-split. Since all Levi subgroups $M \subset G$ are products of general linear groups with (quasi-split)

[32] Pure inner forms of $\mathrm{SO}(V')$ are isomorphic to $\mathrm{SO}(V)$ with V of the same discriminant and dimension as V', see [PR94, Proposition 2.8 on page 73]. The fact that an order 3 automorphism cannot occur follows from [PR94, Proposition 2.20 on page 90].

special orthogonal groups of even rank, we in fact have maps $\widetilde{\mathrm{LL}}_M$ for all $M \subset G$. These satisfy the obvious analogue of Assumption (1) by [Pen25b, Proposition 2.4.3], and the obvious analogue of Assumption (2) by work of Mœglin [Mœg11, Theorem 1.5.1], cf. [Xu17, Theorem 3.3], see [Pen25b, Corollary 2.5.2]. We can prove the obvious analogue of Assumption (3) by a globalization argument as in the proof of Theorem 7.3.8, using Corollary 6.4.5. The globalization argument there can be adapted as follows: The quadratic extension K/F over which G^* becomes split (corresponding to the discriminant of the quadratic space) should be globalized to a totally real[33] extension K/L^+, and we should choose H^* to be an even special orthogonal group over L^+ such that $\mathrm{Gal}(\mathsf{K}/\mathsf{L}^+)$ acts nontrivially on its root datum (for example by choosing a quadratic space with discriminant corresponding to K). Then H^* is split over $\mathbb{R}$ and we can choose an appropriate inner form H as in the proof of Lemma 7.3.4 (thus in particular, we will globalize to type $D^{\mathbb{R}}$). To globalize, we will use [CZ25, Lemma 6.14] together with the existence of supercuspidal representations satisfying **ssc** for an auxiliary place $w \nmid 2$ of L^+ which splits in K, see [AHKO25, Theorem 1.1].[34] Finally, we note that Lemma 7.3.6 generalizes to the case of even special orthogonal groups with the same proof (keeping in mind Section 6.3.3).

Theorem 7.4.1 in the case that G is quasi-split then follows verbatim from the proof of Proposition 7.2.7 using Corollary 6.4.5, once we can establish the constancy of $\widetilde{\mathrm{LL}}_G^{\mathrm{FS}}$ on the L-packets $\widetilde{\Pi}_\phi(G)$ for all even special orthogonal groups G. For this, write $G' \supset G$ for the orthogonal group containing the special orthogonal group G. Note that $G'(F) \supsetneq G(F)$ is a proper inclusion because reflections have determinant -1. We have the following analogue of Assumption (4), which holds by [Art13], see [Var24, Proposition 5.1.2, Remark 2.5.10]: For $\phi \in \widetilde{\Phi}_{\mathrm{temp}}(G)$, there is a stable virtual character

$$\Theta_\phi^1 = \sum_{\pi \in \widetilde{\Pi}_\phi(G)} a_\pi \Theta_\pi,$$

that is $G'(F)$-invariant with all coefficients $a_\pi \neq 0$, and moreover atomic with respect to this property. By [Han26, Theorem 1.1] the Fargues–Scholze map factors as

$$\Psi_G^{\mathrm{FS}} \colon \mathcal{Z}^{\mathrm{spec}}(G) \to \mathcal{Z}^{\mathrm{vst}}(G) \subseteq \mathcal{Z}(G).$$

On the other hand, by functoriality of Ψ_G^{FS} under isomorphisms, it is naturally $G'(F)$-equivariant, hence induces

$$\Psi_G^{\mathrm{FS}} \colon \widetilde{\mathcal{Z}}^{\mathrm{spec}}(G) \to \mathcal{Z}^{\mathrm{vst}}(G)^{G'(F)}.$$

This means that for every $z \in \widetilde{\mathcal{Z}}^{\mathrm{spec}}(G)$ the convolution $z * \Theta_\phi^1$ is a linear combination of Θ_π for $\pi \in \widetilde{\Pi}_\phi(G)$ that is stable and $G'(F)$-invariant, and hence a constant multiple of Θ_ϕ^1. This implies that $\widetilde{\mathrm{LL}}_G^{\mathrm{FS}}$ is constant on $\widetilde{\Pi}_\phi(G)$.

[33] So that we get a Shimura datum of type $D^{\mathbb{R}}$.

[34] More precisely, they show the existence of irreducible $2n$-dimensional orthogonal representations of W_F, which correspond under LLC to the desired supercuspidal representations.

To deal with non quasi-split G, we can combine the above argument with the proof of Proposition 7.2.9, if we can prove that for all matching functions $f^* \in \mathcal{C}_c^\infty(G^*(F))$, $f \in \mathcal{C}_c^\infty(G(F))$, there is an equality

$$\Theta_\phi^{\mathbf{1}}(f^*) = \sum_{\pi \in \widetilde{\Pi}_\phi(G)} c_\pi \Theta_\pi(f),$$

for some coefficients $c_\pi \in \mathbb{C}^\times$ (depending on $\mathfrak{w}$, but not on f, f^*). This holds by [Pen25a, Theorem A]. □

We can now prove Theorem II.

Proof of Theorem II. By Lemma 7.1.4, it suffices to check the compatibility for tempered irreducible smooth representations π. In this case, it follows from Theorem 7.3.8 and Theorem 7.4.1. □

7.4.3. We have the following analogue of [Pen25b, Theorem 7.1.1], which is Corollary III. Let F be a p-adic local field as before. Let V' be a $2n$-dimensional quadratic space over F with quasi-split special orthogonal group $G^* = \mathrm{SO}(V)$, and let G be a pure inner form of G^*.

Theorem 7.4.4. *There is a unique local Langlands correspondence* $\mathrm{LL}_G : \Pi_{\mathrm{temp}}(G) \to \Phi_{\mathrm{temp}}(G)$ *lifting* $\widetilde{\mathrm{LL}}_G$ *such that* $(-)^{\mathrm{ss}} \circ \mathrm{LL}_G = \mathrm{LL}_G^{\mathrm{FS}}$. *Moreover, it satisfies the conditions listed in* [Pen25b, Theorem 7.1.1].

Proof. Given Theorem 7.4.1, the proof of [Pen25b, Theorem 7.1.1] goes through verbatim. □

Note that Theorem 7.4.4 is proved in [Pen25b] if $p > 2$ and F is unramified, see [Pen25b, Theorem 7.1.1].

Appendix A. Canonical Frobenius descents revisited

The goal of this appendix is to extend [DvHKZ24a, Section 8.2] regarding canonical Frobenius descent data, using the theory of six functor formalisms of [Man22, HM24] and results of [DK25]. The arguments here were suggested to us by Lucas Mann, although all mistakes should be attributed to us. In this appendix we will use the word ordinary category to mean category, the word category to mean $(\infty, 1)$-category, and the word symmetric monoidal category to mean symmetric monoidal $(\infty, 1)$-category. Keeping this convention in mind, we let Cat denote the category of categories, see [HM24, Example D.1.4]. We will let $\mathrm{Cat}^\times$ denote the symmetric monoidal category given by the cartesian monoidal structure, see [HM24, Example B.1.9].

A.1. We consider vStk, the category of small v-stacks. Let Λ be a $\mathbb{Z}_\ell$-algebra in which ℓ is nilpotent. By [Man26, Proposition 3.16], there is a functor

$$\mathcal{D}^* \colon \mathrm{vStk}^{\mathrm{op}} \to \mathrm{Cat}, \quad X \mapsto \mathcal{D}_{\acute{e}t}(X, \Lambda),$$

sending $f\colon X \to Y$ to $f^*\colon \mathcal{D}_{\text{ét}}(Y,\Lambda) \to \mathcal{D}_{\text{ét}}(X,\Lambda)$. There is a natural transformation $1_{\mathrm{vStk}} \to 1_{\mathrm{vStk}}$ sending X to the morphism $\phi\colon X \to X$. This induces a natural transformation $\mathcal{D}^*(\phi)\colon \mathcal{D}^* \to \mathcal{D}^*$.

Lemma A.2. *There is an isomorphism $\mathcal{D}^*(\phi) \simeq \mathrm{id}_{\mathcal{D}^*}$ of natural transformations.*

Proof. Let StrTotDisc $\subset$ vStk be the full subcategory of strictly totally disconnected affinoid perfectoid spaces. Consider $\mathrm{HShv}(\mathrm{vStk},\mathrm{Cat}) \subset \mathrm{Fun}(\mathrm{vStk}^{\mathrm{op}},\mathrm{Cat})$ the full subcategory of hypersheaves for the v-topology (in the ∞-categorical sense). Then by [Man26, Proposition 3.16], we have that $\mathcal{D}^*$ lies in this full subcategory. By hyperdescent, the restriction functor

$$\mathrm{HShv}(\mathrm{vStk},\mathrm{Cat}) \xrightarrow{\sim} \mathrm{HShv}(\mathrm{StrTotDisc},\mathrm{Cat})$$

is an equivalence as every v-stack admits a v-hypercover by strictly totally disconnected perfectoid spaces, and so it suffices to construct $\mathcal{D}^*(\phi) \simeq \mathrm{id}_{\mathcal{D}^*}$ on this subcategory. We now note that if $X \in \mathrm{StrTotDisc}$ then $\phi = \mathrm{id}$ as maps on underlying topological spaces $|X| \to |X|$. On the other hand, $\mathcal{D}_{\text{ét}}(X,\Lambda) = \mathcal{D}(\mathrm{Sh}(|X|,\Lambda))$ by construction, see [Sch22, Remark 14.14]. This shows that there is a natural equivalence $\phi^* \simeq \mathrm{id}$ of functors $\mathcal{D}_{\text{ét}}(X,\Lambda) \to \mathcal{D}_{\text{ét}}(X,\Lambda)$, functorial in $X \in \mathrm{StrTotDisc}$. □

A.3. There is a functor $\mathcal{D}^{\phi,*}\colon \mathrm{vStk}^{\mathrm{op}} \to \mathrm{Cat}$ sending X to the category $\mathcal{D}([X/\phi^{\mathbb{Z}}])$. Indeed, we are just precomposing $\mathcal{D}^*$ with $\mathrm{vStk}^{\mathrm{op}} \to \mathrm{vStk}^{\mathrm{op}}$ given by $X \mapsto [X/\phi^{\mathbb{Z}}]$. Let $B\mathbb{Z}$ be the group of integers considered as a category with one object. There is a functor $\mathrm{Cat} \to \mathrm{Cat}$ sending $\mathcal{E} \mapsto \mathrm{Fun}(B\mathbb{Z},\mathcal{E}) \coloneqq \mathcal{E}^{B\mathbb{Z}}$. Let us write $\mathcal{D}^{B\mathbb{Z},*}\colon \mathrm{vStk}^{\mathrm{op}} \to \mathrm{Cat}$ for the composition of $\mathcal{D}^*$ with this functor.

Lemma A.4. *There is an isomorphism of functors $\mathcal{D}^{\phi,*} \xrightarrow{\sim} \mathcal{D}^{B\mathbb{Z},*}$.*

Proof. We note that by descent (see [DvHKZ24a, Lemma 8.2.7]), there is a canonical equivalence

$$\mathcal{D}([X/\phi^{\mathbb{Z}}]) \simeq \mathcal{D}(X)^{B\phi^{\mathbb{Z}}},$$

where the right hand side is the category $\mathbb{Z}$-equivariant objects with $\mathbb{Z}$ acting via ϕ^* on $\mathcal{D}(X)$. Applying Lemma A.2 to functorially identify the action of ϕ^* with the trivial action, we deduce that $\mathcal{D}^{\phi,*}$ is isomorphic to $\mathcal{D}^{B\mathbb{Z},*}$. □

A.5. Recall the notion of geometric setup from [HM24, Definition 2.1.1]. We consider the geometric setup $(\mathcal{C},E) = (\mathrm{vStk}^{\mathrm{op}}, \ell{-}\mathrm{fine})$, where $\ell-$fine is the set of ℓ-fine maps between small v-stacks in the sense of [Man26, Definition 5.8]. Note that there is a natural morphism of geometric setups

$$\alpha : (\mathcal{C},E) \to (\mathcal{C},E), \quad X \mapsto [X/\phi^{\mathbb{Z}}]$$

that follows from étale descent for ℓ-fine maps, see [Man26, Lemma 5.10].

A.6. Associated to $(\mathcal{C}, E)$ is the symmetric monoidal category of correspondences $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, see [HM24, Definition 2.2.10], whose underlying category receives a functor $h\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, E)$, see [HM24, Remark 2.2.12]. Recall the notion of 6-functor formalism from [HM24, Definition 3.2.1]: In our setting these are 3-functor formalisms, that is, lax symmetric monoidal functors, satisfying certain properties. By [Man26, Theorem 5.11], there is a 6-functor formalism

$$\mathcal{D}\colon\ \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$$

whose restriction along h is isomorphic to the functor $\mathcal{D}^*$ discussed above. We similarly have $\mathcal{D}^{\phi}$, which is the composition of the morphism of geometric setups $\alpha\colon (\mathcal{C}, E) \to (\mathcal{C}, E)$ with the six functor formalism $\mathcal{D}$, and thus itself a 6-functor formalism. We also consider the 3-functor formalism $\mathcal{D}^{B\mathbb{Z}} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ given by composing $\mathcal{D}$ with the symmetric monoidal functor $\mathcal{E} \mapsto \mathcal{E}^{B\mathbb{Z}}$.

Proposition A.7. *There is an isomorphism of* 3*-functor formalisms*

$$\mathcal{D}^{\phi} \xrightarrow{\sim} \mathcal{D}^{B\mathbb{Z}}.$$

Because being a 6*-functor formalism is a property, see* [HM24, Definition 3.2.1]*, it is in particular an isomorphism of* 6*-functor formalisms.*

Proof. As in the proof of [Man26, Proposition 5.6], we consider the auxiliary collections of edges (in the notation of loc. cit.) $E_1 = \mathrm{fdcqc} \supset E_2 = \mathrm{fdcsqc} \subset E_3 = \mathrm{fdcss} \subset E_4 = \mathrm{fdcs} \subset E_5 = E$. We also consider the full subcategories $\mathcal{C}_i \subset \mathcal{C}$ which equals $\mathcal{C}$ unless $i = 3$, in which case it equals the category of separated locally spatial diamonds. For $i = 1, \ldots, 5$, we write $\mathcal{D}_i^{\phi}, \mathcal{D}_i^{B\mathbb{Z}}$ for the restriction of $\mathcal{D}^{\phi}, \mathcal{D}^{B\mathbb{Z}}$ to $\mathrm{Corr}(\mathcal{C}_i, E_i)$. As in [Man26, proof of Proposition 5.6], we define $I \subset E_1$ to be the class of open immersions and $P \subset E_1$ to be the class of proper fdcqc maps.

It is explained in loc. cit. that the pair $I, P \subset E_1$ is a suitable decomposition as in [Man22, Definition A.5.9], which implies that $(\mathcal{C}_1, E_1, I, P)$ is a Nagata setup in the sense of [DK25, Definition 2.1]. It now follows from [DK25, Theorem 3.3] that the 3-functor formalisms $\mathcal{D}_1^{\phi}, \mathcal{D}_1^{B\mathbb{Z}}$ are uniquely determined by $\mathcal{D}^{\phi,*}$ and $\mathcal{D}^{B\mathbb{Z},*}$, and thus the isomorphism of Lemma A.4 induces an isomorphism of 3-functor formalisms $\mathcal{D}_1^{\phi} \xrightarrow{\sim} \mathcal{D}_1^{B\mathbb{Z}}$.[35] Here we are using that morphisms in P are cohomologically proper in $\mathcal{D}_1^{\phi}, \mathcal{D}_1^{B\mathbb{Z}}$, see [Man26, Lemma 9.8], and morphisms in I are cohomologically étale in $\mathcal{D}_1^{\phi}, \mathcal{D}_1^{B\mathbb{Z}}$, see [Man26, Proposition 5.6.(ii)], to verify the hypotheses of [DK25, Theorem 3.3].

We now take the isomorphism of 3-functor formalisms $\mathcal{D}_1^{\phi} \xrightarrow{\sim} \mathcal{D}_1^{B\mathbb{Z}}$ and show that it induces isomorphisms of 3-functor formalisms $\mathcal{D}_i^{\phi} \xrightarrow{\sim} \mathcal{D}_i^{B\mathbb{Z}}$ for $i = 2, 3, 4, 5$. Note that going from $i = 1$ to $i = 2$ is vacuous since $E_2 \subset E_1$. Going from $i = 2$ to $i = 3$ follows from the uniqueness statements of [Man22, Proposition A.5.12, A.5.14]; the assumptions of these propositions hold as explained in the proof of [Man22, Proposition 5.6]. Going from $i = 3$ to $i = 4$ follows directly from (the

[35] Note that the isomorphism of Lemma A.4 is automatically an isomorphism of monoidal functors, because we are using the Cartesian monoidal structure.

uniqueness part of) [HM24, Proposition 3.4.2], because E_4 consists precisely of edges representable in E_3. To go from $i = 4$ to $i = 5$, we note that the notion of ℓ-fine morphisms $f : X \to Y$ agrees with the notion of morphisms that are $\mathcal{D}^!$-locally on the source fdcs, in the sense of [HM24, Definition 3.4.10], which is precisely condition (ii) of [HM24, Proposition 3.4.8]. Thus we can go from $i = 4$ to $i = 5$ using the uniqueness part of [HM24, Proposition 3.4.8]. □

A.8. The natural maps $X \to [X/\phi^{\mathbb{Z}}]$ induce a morphism $\mathcal{D}^\phi \to \mathcal{D}$ of 3-functor formalisms, which corresponds to "forgetting the ϕ-descent datum". Under the isomorphism of Proposition A.7, this corresponds to the "forgetful map" $\mathcal{D}^{B\mathbb{Z}} \to \mathcal{D}$. The map $\mathcal{D}^{B\mathbb{Z}} \to \mathcal{D}$ has a section corresponding to "taking the trivial $\mathbb{Z}$-action", which under our identification induces a section $\mathrm{can}\colon \mathcal{D} \to \mathcal{D}^\phi$ of 3-functor formalisms corresponding to "taking the canonical ϕ-descent datum": This is the canonical ϕ-descent of [DvHKZ24a, Lemma 8.2.4].

Proposition A.9. *Let $f\colon X \to Y$ be a morphism of small v-stacks and consider the associated induced map of v-stacks*

$$f_\phi\colon [X/\phi_X^{\mathbb{Z}}] \to [Y/\phi_Y^{\mathbb{Z}}].$$

(*i*) *There is a natural isomorphism* $\mathrm{can}_Y \circ Rf_* \xrightarrow{\sim} Rf_{\phi,*} \circ \mathrm{can}_X$ *of functors* $\mathcal{D}(X) \to \mathcal{D}([Y/\phi^{\mathbb{Z}}])$.

(*ii*) *If f is ℓ-fine, then there is a natural isomorphism* $\mathrm{can}_Y \circ f_! \xrightarrow{\sim} f_{\phi,!} \circ \mathrm{can}_X$ *of functors* $\mathcal{D}(X) \to \mathcal{D}([Y/\phi^{\mathbb{Z}}])$.

Proof. By Proposition A.7, we may compute $Rf_{\phi,*}$ and $f_{\phi,!}$ using the $\mathcal{D}^{B\mathbb{Z}}$-theory instead of the $\mathcal{D}^\phi$-theory. That is, it suffices to produce natural isomorphisms

$$Rf_{B\mathbb{Z},*}(A, \mathrm{id}_A) \cong (Rf_*A, \mathrm{id}_{Rf_*A}), \quad f_{B\mathbb{Z},!}(A, \mathrm{id}_A) \cong (f_!A, \mathrm{id}_{f_!A}),$$

where by $Rf_{B\mathbb{Z},*}$ and $f_{B\mathbb{Z},!}$ we mean the corresponding functors extracted from $\mathcal{D}^{B\mathbb{Z}}\colon \mathrm{Corr}(\mathcal{C}, E)^\otimes \to \mathrm{Cat}^\times$ as in [HM24, Definition 3.1.4, 3.2.1].

For $f_!$, recall from Appendix A.3 that the 3-functor formalism $\mathcal{D}^{B\mathbb{Z}}$ is simply defined as a composition of $\mathcal{D}$ with the endofunctor $(-)^{B\mathbb{Z}}\colon \mathrm{Cat} \to \mathrm{Cat}$. This endofunctor sends a functor $F\colon \mathcal{E} \to \mathcal{F}$ to $(A, \alpha \in \mathrm{Aut}(A)) \mapsto (F(A), F(\alpha))$, and therefore

$$f_{B\mathbb{Z},!}(A, \mathrm{id}_A) = (f_!A, f_!\mathrm{id}_A) = (f_!A, \mathrm{id}_{f_!A}).$$

Similarly, we see that

$$f^{B\mathbb{Z},*}(A, \alpha) = (f^*A, f^*\alpha).$$

To access $f_{B\mathbb{Z},*}$, we need to compute the right adjoint of $f^{B\mathbb{Z},*}$. By Lemma A.10, the right adjoint of $f^{B\mathbb{Z},*} = (f^*)^{B\mathbb{Z}}$ is given by $(Rf_*)^{B\mathbb{Z}}$, and therefore

$$Rf_{B\mathbb{Z},*}(A, \alpha) = (Rf_*)^{B\mathbb{Z}}(A, \alpha) = (Rf_*A, Rf_*\alpha).$$

Applying this to $\alpha = \mathrm{id}_A$, we obtain the desired identification. □

Lemma A.10. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of categories with right adjoint G, and let K be a simplicial set. Then $F^K\colon \mathcal{C}^K \to \mathcal{D}^K$ is left adjoint to G^K.*

Proof. As we see from [Lur09, Definition 5.2.2.1], we obtain an inner fibration $p\colon \mathcal{M} \to \Delta^1$ that is both Cartesian and coCartesian, together with equivalences $\mathcal{C} \to p^{-1}\{0\}$ and $\mathcal{D} \to p^{-1}\{1\}$. By [Lur09, Proposition 3.1.2.1], the projection

$$q\colon \mathcal{M}^K \times_{(\Delta^1)^K} \Delta^1 \to \Delta^1$$

is both Cartesian and coCartesian, and moreover q-Cartesian edges are precisely those $K \times \Delta^1 \to \mathcal{M}$ whose restriction to each $\{k\} \times \Delta^1$ is p-Cartesian and similarly for q-coCartesian edges. This shows that both F^K and G^K are associated to q. $\square$

A.11. Now let L be a finite extension of $\mathbb{Q}_p$. Let C be a completed algebraic closure of L and $\breve{L} \subset C$ the completion of the maximal unramified extension. Let $k_L \subset k$ be the extension on residue fields induced by $L \subset \breve{L}$, and let $r = [k_L : \mathbb{F}_p]$. As in [DvHKZ24a, Section 8.2.13], we consider the action of $\underline{\mathrm{Gal}_L} \times \underline{\mathbb{Z}}$ on $\operatorname{Spd} C$, where $\underline{\mathbb{Z}}$ acts by ϕ_C. We also consider the Weil group $W_L \subset \mathrm{Gal}_L$, which admits a twisted embedding $\iota : W_L \to \mathrm{Gal}_L \times \mathbb{Z}$. We consider (see [DvHKZ24a, Lemma 8.2.12])

$$\operatorname{Spd} C/\underline{\iota(W_L)} \xrightarrow{\sim} \left[\operatorname{Spd} L/\phi_L^{r\mathbb{Z}}\right] \times_{\operatorname{Spd} k_L} \operatorname{Spd} k.$$

A.12. For any small v-stack X we can consider $\mathcal{D}(X)$ as a condensed ∞-category by considering the functor taking a profinite set S to $\mathcal{D}(X \times \underline{S})$. This allows us to consider $\mathcal{D}(X)^{BH}$ for a locally profinite group H, see [DvHKZ24a, Section 8.2.8]. By [DvHKZ24a, Lemma 8.2.10], there are natural identifications

$$\mathcal{D}(\operatorname{Spd} L) \xrightarrow{\sim} \mathcal{D}(\operatorname{Spd} C)^{B\,\mathrm{Gal}_L}$$
$$\mathcal{D}(\operatorname{Spd} C/\underline{\iota(W_L)}) \xrightarrow{\sim} \mathcal{D}(\operatorname{Spd} C)^{BW_L},$$

and so restricting along $W_L \to \mathrm{Gal}_L$ defines a natural map

$$\rho : \mathcal{D}(\operatorname{Spd} L) \to \mathcal{D}(\operatorname{Spd} C/\underline{\iota(W_L)}).$$

A.13. Let $f : X \to \operatorname{Spd} L$ be a morphism of small v-stacks, and consider the morphism

$$f_{\phi,k} : \left[X/\phi_X^{r\mathbb{Z}}\right] \times_{\operatorname{Spd} k_L} \operatorname{Spd} k \to \left[\operatorname{Spd} L/\phi_L^{r\mathbb{Z}}\right] \times_{\operatorname{Spd} k_L} \operatorname{Spd} k,$$

note that it is ℓ-fine if f is, by étale descent for ℓ-fine morphisms. Note that the map $f_{\phi,k}$ is the base change of f_ϕ to k over $\operatorname{Spd} \mathbb{F}_p$.

Proposition A.14. *Let f and $f_{\phi,k}$ be as above, and consider the natural projection map $\psi\colon [X/\phi_X^{\mathbb{Z}}] \times \operatorname{Spd} k \to [X/\phi_X^{\mathbb{Z}}]$.*

- *(i) If f is qcqs, then there is a natural isomorphism $\rho \circ Rf_* \simeq Rf_{\phi,k,*} \circ \psi^* \circ \mathrm{can}_X$ of functors $\mathcal{D}^+(X) \to \mathcal{D}^+([\operatorname{Spd} C/\underline{\iota(W_L)}])$.*
- *(ii) If f is ℓ-fine, then there is a natural isomorphism $\rho \circ f_! \simeq f_{\phi,k,!} \circ \psi^* \circ \mathrm{can}_X$ of functors $\mathcal{D}(X) \to \mathcal{D}([\operatorname{Spd} C/\underline{\iota(W_L)}])$.*
- *(iii) If X is the analytification of a smooth variety X^{alg}/L and if A is the analytification of a locally constant sheaf of Λ-modules A^{alg} on $X^{\mathrm{alg}}_{\text{ét}}$, then there is a natural isomorphism $\rho(Rf_*A) \cong Rf_{\phi,k,*}\psi^*\mathrm{can}_X(A)$.*

Proof. The proof is the same as [DvHKZ24a, Proposition 8.2.15]. We consider the cartesian diagram

$$\begin{tikzcd} {[X/\phi^{\mathbb{Z}}] \times \operatorname{Spd} k} \arrow[r, "\psi"] \arrow[d, "f_{\phi,k}"] & {[X/\phi^{\mathbb{Z}}]} \arrow[d, "f_\phi"] \\ {[\operatorname{Spd} L/\phi^{\mathbb{Z}}] \times \operatorname{Spd} k = [\operatorname{Spd} C/\underline{\iota(W_L)}]} \arrow[r, "g"] & {[\operatorname{Spd} L/\phi^{\mathbb{Z}}] = [\operatorname{Spd} C/(\underline{\mathrm{Gal}_L} \times \underline{\mathbb{Z}})],} \end{tikzcd}$$

and apply Proposition A.9 to identify $Rf_{\phi,*} \circ \mathrm{can}_X \simeq \mathrm{can}_L \circ Rf_*$ and similarly $f_{\phi,!} \circ \mathrm{can}_X \simeq \mathrm{can}_L \circ f_!$ for (ii). We then use qcqs base change [Sch22, Proposition 17.6] to identify $Rf_{\phi,k,*}\psi^* \cong g^* Rf_{\phi,*}$ for (i) and proper base change [Sch22, Proposition 22.19] to identify $f_{\phi,k,!}\psi^* \cong g^* f_{\phi,!}$ for (ii). Finally, we note that $\rho = g^* \circ \mathrm{can}$, completing the proof.

For (iii), it remains to verify that the base change morphism

$$g^* Rf_{\phi,*}A_\phi \to Rf_{\phi,k,*}\psi^* A_\phi$$

is an isomorphism, where $A_\phi \in \mathcal{D}([X/\phi^{\mathbb{Z}}])$ is the canonical descent of A. As isomorphisms can be detected on a v-cover, we may use smooth base change [Sch22, Proposition 23.16.(ii)] along the separated étale maps $\operatorname{Spd} \breve{L} \to [\operatorname{Spd} L/\phi^{\mathbb{Z}}] \times \operatorname{Spd} k$ and $\operatorname{Spd} L \to [\operatorname{Spd} L/\phi^{\mathbb{Z}}]$ to reduce to checking that the base change morphism

$$\tilde{g}^* Rf_* A \to Rf_{\breve{L}*}\tilde{\psi}^* A$$

is an isomorphism in the diagram

$$\begin{tikzcd} X_{\breve{L}} \arrow[r, "\tilde{\psi}"] \arrow[d, "f_{\breve{L}}"'] & X \arrow[d, "f"] \\ \operatorname{Spd} \breve{L} \arrow[r, "\tilde{g}"] & \operatorname{Spd} L. \end{tikzcd}$$

This follows from comparison with the algebraic theory. By Lemma A.15 below, we may instead verify that $\tilde{g}^{\mathrm{alg},*} Rf_*^{\mathrm{alg}} A^{\mathrm{alg}} \cong Rf_{\breve{L},*}^{\mathrm{alg}} \tilde{\psi}^{\mathrm{alg},*} A^{\mathrm{alg}}$ as étale sheaves on $\operatorname{Spec} \breve{L}$. This is now [SGA73, Corollary XVI.1.6]. $\square$

Lemma A.15. *Let L be a complete nonarchimedean discretely valued field extension of $\mathbb{Q}_p$, and let $f\colon X \to \operatorname{Spec} L$ be a smooth separated algebraic variety with analytification $f^\diamondsuit\colon X^\diamondsuit \to \operatorname{Spd} L$. Let Λ be a ring in which ℓ is nilpotent, and let A be a locally constant sheaf of Λ-modules on the étale site of X. Then*

$$(Rf_*A)^\diamondsuit \xrightarrow{\sim} Rf_*^\diamondsuit A^\diamondsuit$$

Proof. Because the comparison theorem [Sch22, Proposition 27.5] is only stated for constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, we make the following roundabout argument. First we reduce to the case when A is trivial. We may as well assume that X is connected. Because A comes from a locally constant sheaf on X, there is a finite Galois étale cover $\pi\colon \tilde{X} \to X$ with Galois group G for which $\pi^*A = M$ is a constant sheaf. Then Rf_*A is the derived G-fixed points of $R(f \circ \pi)_*M$, see [LZ24, Corollary 6.2.14], and similarly for $Rf_*^\diamondsuit$. Therefore we may replace (X, A) with $(\tilde{X}, M)$ and assume that A is trivial.

We now use Verdier duality, [Sch22, Theorem 23.3.(i)] and [LZ24, Proposition 6.2.4.(4)]. Choose n large enough so that $\ell^n\Lambda = 0$. Since M is constant, we can write $M = f^*N$. Then $M = R\mathscr{H}om_{\mathbb{Z}/\ell^n\mathbb{Z}}(\mathbb{Z}/\ell^n\mathbb{Z}, Rf^!N(-d)[-2d])$ both algebraically and analytically, and it thus suffices to show that

$$R\mathscr{H}om_{\mathbb{Z}/\ell^n\mathbb{Z}}(Rf_!(\mathbb{Z}/\ell^n\mathbb{Z}), N(-d)[-2d])^\diamondsuit \cong R\mathscr{H}om_{\mathbb{Z}/\ell^n\mathbb{Z}}(Rf_!^\diamondsuit(\mathbb{Z}/\ell^n\mathbb{Z}), N^\diamondsuit(-d)[-2d]).$$

Since analytification induces an equivalence $\mathcal{D}^+(\operatorname{Spec} L, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathcal{D}^+(\operatorname{Spd} L, \mathbb{Z}/\ell^n\mathbb{Z})$ by [Sch22, Proposition 14.15] applied to $Y = \operatorname{Spd} L$, whose étale site agrees with the étale site of $\operatorname{Spa} L$ by [Sch22, Lemma 15.6] and thus with the étale site of $\operatorname{Spec} L$, it is enough to show that $(Rf_!(\mathbb{Z}/\ell^n\mathbb{Z}))^\diamondsuit \cong Rf_!^\diamondsuit(\mathbb{Z}/\ell^n\mathbb{Z})$. This now follows from [Sch22, Proposition 27.5]. □

References

[ABLB+25] Johannes Anschütz, Guido Bosco, Arthur-César Le Bras, Juan Esteban Rodríguez Camargo, and Peter Scholze, *Analytic de Rham stacks of Fargues-Fontaine curves*, arxiv 2510.15196, 2025. 9

[AGI+24] Hiraku Atobe, Wee Teck Gan, Atsushi Ichino, Tasho Kaletha, Alberto Mínguez, and Sug Woo Shin, *Local intertwining relations and co-tempered a-packets of classical groups*, arxiv 2410.13504, October 2024. 4, 70, 80

[AHKO25] Moshe Adrian, Guy Henniart, Eyal Kaplan, and Masao Oi, *Simple supercuspidal L-packets of split special orthogonal groups over dyadic fields*, J. Lond. Math. Soc. (2) **112** (2025), no. 1, Paper No. e70223, 62. 85

[Ans23] Johannes Anschütz, *G-bundles on the absolute Fargues-Fontaine curve*, Acta Arith. **207** (2023), no. 4, 351–363. 18

[Aok26] Ko Aoki, *Berkovich 2-motives and normed ring stacks*, arxiv 2603.01877, 2026. 9

[Art13] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013, Orthogonal and symplectic groups. 2, 4, 5, 6, 69, 70, 72, 80, 81, 82, 84, 85

[BG14] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187. 58, 68

[BMHN24] Alexander Bertoloni Meli, Linus Hamann, and Kieu Hieu Nguyen, *Compatibility of the Fargues-Scholze correspondence for unitary groups*, Math. Ann. **390** (2024), no. 3, 4729–4787. 2, 5, 6, 12, 72

[BMR05] Michael Bate, Benjamin Martin, and Gerhard Röhrle, *A geometric approach to complete reducibility*, Invent. Math. **161** (2005), no. 1, 177–218. 65

[BR94] Don Blasius and Jonathan D. Rogawski, *Zeta functions of Shimura varieties*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Part 2, Amer. Math. Soc., Providence, RI, 1994, pp. 525–571. 3, 7, 8

[Bre94] Lawrence Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque (1994), no. 225, 160. 46

[BS15] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, Astérisque (2015), no. 369, 99–201. 34

[BW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. 65

[Car12] Ana Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413. 70

[CG15] Ping-Shun Chan and Wee Teck Gan, *The local Langlands conjecture for* GSp(4) *III: Stability and twisted endoscopy*, J. Number Theory **146** (2015), 69–133. 81

[CH13] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. **1** (2013), no. 1, 53–73. 70

[Che51] Claude Chevalley, *Deux théorèmes d'arithmétique*, J. Math. Soc. Japan **3** (1951), 36–44. 17

[CvdHS25] Robert Cass, Thibaud van den Hove, and Jakob Scholbach, *The geometric Satake equivalence for integral motives*, Compos. Math. **161** (2025), no. 11, 2755–2851. 77

[Clo90] Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159. 70

[CS17] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Ann. of Math. (2) **186** (2017), no. 3, 649–766. 12

[CZ21] Rui Chen and Jialiang Zou, *Local Langlands correspondence for even orthogonal groups via theta lifts*, Selecta Math. (N.S.) **27** (2021), no. 5, Paper No. 88, 71. 2, 5, 69, 84

[CZ25] ______, *Arthur's multiplicity formula for even orthogonal and unitary groups*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 12, 4769–4843. 70, 85

[Dan25] Patrick Daniels, *Canonical integral models for Shimura varieties of toral type*, Algebra Number Theory **19** (2025), no. 2, 247–286. 16

[Del73] Pierre Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., vol. Vol. 349, Springer, Berlin-New York, 1973, pp. 501–597. 74

[Del79] ______, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289. 9, 10, 33, 34, 35, 66

[DHKM24a] Jean-François Dat, David Helm, Robert Kurinczuk, and Gilbert Moss, *Finiteness for Hecke algebras of p-adic groups*, J. Amer. Math. Soc. **37** (2024), no. 3, 929–949. 77

[DHKM24b] ______, *Local Langlands in families: The banal case*, arxiv 2406.09283, June 2024. 61, 81

[DHKM25] ______, *Moduli of Langlands parameters*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 5, 1827–1927. 58

[DvHKZ24a] Patrick Daniels, Pol van Hoften, Dongryul Kim, and Mingjia Zhang, *Igusa stacks and the cohomology of Shimura varieties*, arxiv 2408.01348, August 2024. 2, 3, 8, 9, 10, 11, 12, 13, 16, 19, 43, 45, 46, 49, 50, 51, 52, 55, 56, 57, 58, 59, 60, 61, 62, 86, 87, 89, 90, 91

[DvHKZ24b] ______, *On a conjecture of Pappas and Rapoport*, arxiv 2403.19771, 2024. 16

[DK25] Adam Dauser and Josefien Kuijper, *Uniqueness of six-functor formalisms*, arxiv 2412.15780, 2025. 86, 88

[DR09] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901. 76

[DY25] Patrick Daniels and Alexander Youcis, *Canonical integral models for Shimura varieties of abelian type*, Forum Math. Sigma **13** (2025), Paper No. e69, 47. 16, 52, 53

[Far20] Laurent Fargues, *Simple connexité des fibres d'une application d'Abel-Jacobi et corps de classes local*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 1, 89–124. 25

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. 68

[FS24] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, arxiv 2102.13459, 2024. 4, 5, 19, 24, 49, 51, 55, 57, 58, 59, 60, 72, 75, 76, 77, 78, 84

[GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, no. 346, 2012, Sur les conjectures de Gross et Prasad. I, pp. 1–109. 7, 68, 71, 83

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 179, Springer-Verlag, Berlin-New York, 1971. 46

[GIZ23] Ian Gleason, Alexander B. Ivanov, and Felix Zillinger, *Meromorphic vector bundles on the Fargues–Fontaine curve*, arxiv 2307.00887, July 2023. 11

[GL22] Ian Gleason and João Lourenço, *On the connectedness of p-adic period domains*, arxiv 2210.08625, October 2022, p. 2210.08625. 50

[GT11a] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2) **173** (2011), no. 3, 1841–1882. 2, 4, 69, 72, 81, 83

[GT11b] ———, *Theta correspondences for* $\mathrm{GSp}(4)$, Represent. Theory **15** (2011), 670–718. 81

[GT14] Wee Teck Gan and Welly Tantono, *The local Langlands conjecture for* $\mathrm{GSp}(4)$*, II: The case of inner forms*, Amer. J. Math. **136** (2014), no. 3, 761–805. 2, 4, 69, 81

[Hai14] Thomas J. Haines, *The stable Bernstein center and test functions for Shimura varieties*, Automorphic forms and Galois representations. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 415, Cambridge Univ. Press, Cambridge, 2014, pp. 118–186. 78

[Ham25] Linus Hamann, *Compatibility of the Fargues–Scholze and Gan–Takeda local Langlands*, Compos. Math. **161** (2025), no. 9, 2202–2271. 2, 5, 6, 12, 72, 81

[Han16] David Hansen, *p-adic period maps and variations of p-adic Hodge structure*, `http://www.davidrenshawhansen.net/owr-report-hansenaug2016.pdf`, 2016. 13

[Han26] ———, *Excursion operators and the stable Bernstein center*, Forum Math. Pi **14** (2026), e10. 6, 7, 76, 77, 78, 79, 85

[HI25] Linus Hamann and Naoki Imai, *Dualizing complexes on the moduli of parabolic bundles*, J. Reine Angew. Math. **825** (2025), 139–183. 49, 57

[HK25] Paul Hamacher and Wansu Kim, *Point counting on Igusa varieties for function fields*, Adv. Math. **480** (2025), no. part C, Paper No. 110517, 88. 52

[HKW22] David Hansen, Tasho Kaletha, and Jared Weinstein, *On the Kottwitz conjecture for local shtuka spaces*, Forum Math. Pi **10** (2022), Paper No. e13, 79. 5, 47

[HL23] Linus Hamann and Si Ying Lee, *Torsion vanishing for some Shimura varieties*, arxiv 2309.08705, 2023. 2, 9, 50

[HM24] Claudius Heyer and Lucas Mann, *6-functor formalisms and smooth representations*, arxiv 2410.13038, 2024. 86, 87, 88, 89

[vdH24] Thibaud van den Hove, *The stack of spherical Langlands parameters*, arxiv 2409.09522, 2024. 2, 3, 8, 60, 61, 62

[How26] Sean Howe, *Inscription, twistors, and p-adic periods*, arxiv 2508.11589, 2026. 9

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. 70, 76

[HZZ25] Xuhua He, Rong Zhou, and Yihang Zhu, *Stabilizers of irreducible components of affine Deligne-Lusztig varieties*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 6, 2387–2441. 5

[IM20] Naoki Imai and Yoichi Mieda, *Potentially good reduction loci of Shimura varieties*, Tunis. J. Math. **2** (2020), no. 2, 399–454. 13, 31

[Ish24] Hiroshi Ishimoto, *The endoscopic classification of representations of non-quasi-split odd special orthogonal groups*, Int. Math. Res. Not. IMRN (2024), no. 14, 10939–11012. 2, 4, 69, 70, 72, 81, 82

[Kal16] Tasho Kaletha, *The local Langlands conjectures for non-quasi-split groups*, Families of automorphic forms and the trace formula, Simons Symp., Springer, [Cham], 2016, pp. 217–257. 77, 80

[Kim25] Dongryul Kim, *Uniqueness and functoriality of Igusa stacks*, arxiv 2504.15542, April 2025. 2, 3, 10, 11, 13, 14, 33, 36, 37, 40, 41, 43, 44, 45

[Kis10] Mark Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012. 33, 63

[Kis17] ———, mod p *points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914. 12

[KMP16] Wansu Kim and Keerthi Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), Paper No. e28, 34. 63

[KMSW14] Tasho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: Inner forms of unitary groups*, arxiv 1409.3731, 2014. 2, 4, 69, 70, 72, 81, 82

[Kot83] Robert E. Kottwitz, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 289–297. 77

[Kot84] Robert Kottwitz, *Shimura varieties and twisted orbital intergals*, Mathematische Annalen **269** (1984), 287–300. 68

[Kot85] Robert E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220. 18, 22

[Kot86] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. 82

[Kot97] ———, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339. 36

[KP18] Mark Kisin and Georgios Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 121–218. 43

[KPZ24] Mark Kisin, Georgios Pappas, and Rong Zhou, *Integral models of Shimura varieties with parahoric level structure, II*, arxiv 2409.03689, 2024. 52

[KS23a] Arno Kret and Sug Woo Shin, *Galois representations for general symplectic groups*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 1, 75–152. 67, 68, 70, 71, 82

[KS23b] ———, *H^0 of Igusa varieties via automorphic forms*, J. Éc. polytech. Math. **10** (2023), 1299–1390. 72

[KS24] ———, *Galois representations for even general special orthogonal groups*, J. Inst. Math. Jussieu **23** (2024), no. 5, 1959–2050. 66, 67, 70

[KSZ21] Mark Kisin, Sug Woo Shin, and Yihang Zhu, *The stable trace formula for Shimura varieties of abelian type*, arxiv 2110.05381, October 2021. 6, 16, 17, 44, 50, 53

[KZ25] Mark Kisin and Rong Zhou, *Strongly compatible systems associated to semistable abelian varieties*, arxiv 2505.02165, 2025. 5

[Lee22] Si Ying Lee, *Semisimplicity of étale cohomology of certain Shimura varieties*, arxiv 2206.07283, 2022. 3, 8, 62, 64

[LH22] Siyan Daniel Li-Huerta, *The plectic conjecture over local fields*, arxiv 2208.09962, August 2022. 3, 7, 63

[LH26] ———, *Courbes et fibrés vectoriels en théorie de Hodge z-adique globale*, arxiv 2602.04978, 2026. 3, 4

[Lov17a] Tom Lovering, *Filtered F-crystals on Shimura varieties of abelian type*, arxiv 1702.06611, 2017. 42

[Lov17b] ———, *Integral canonical models for automorphic vector bundles of abelian type*, Algebra Number Theory **11** (2017), no. 8, 1837–1890. 9, 42

[LS18] Kai-Wen Lan and Benoît Stroh, *Nearby cycles of automorphic étale sheaves, II*, Cohomology of arithmetic groups, Springer Proc. Math. Stat., vol. 245, Springer, Cham, 2018, pp. 83–106. 56, 63

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387. 24

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. 90

[LZ24] Yifeng Liu and Weizhe Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*, arxiv 1211.5948, 2024. 48, 91, 92

[Man04] Elena Mantovan, *On certain unitary group Shimura varieties*, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, no. 291, Société Mathématique de France, Paris, 2004, pp. 201–331. 8

[Man22] Lucas Mann, *A p-adic 6-functor formalism in rigid-analytic geometry*, arxiv 2206.02022, 2022. 86, 88

[Man26] ———, *A 6-functor formalism for $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves on diamonds*, Res. Math. Sci. **13** (2026), no. 1, Paper No. 19. 46, 47, 48, 49, 50, 51, 54, 86, 87, 88

[Mil05] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 265–378. 27, 30, 66, 71

[Mœg07] Colette Mœglin, *Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques*, Pacific J. Math. **233** (2007), no. 1, 159–204. 81

[Mœg11] ———, *Multiplicité 1 dans les paquets d'Arthur aux places p-adiques*, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 333–374. 81, 85

[Mok15] Chung Pang Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248. 2, 4, 69, 70, 72, 80, 81, 82

[MW26] Shengkai Mao and Peihang Wu, *Canonical extensions of p-adic shtukas on toroidal compactifications of Shimura varieties*, in preparation, 2026. 16

[NS16] J. Nekovář and A. J. Scholl, *Introduction to plectic cohomology*, Advances in the theory of automorphic forms and their L-functions, Contemp. Math., vol. 664, Amer. Math. Soc., Providence, RI, 2016, pp. 321–337. 7

[Oi19a] Masao Oi, *Endoscopic lifting of simple supercuspidal representations of* SO_{2n+1} *to* GL_{2n}, Amer. J. Math. **141** (2019), no. 1, 169–217. 82

[Oi19b] ———, *Endoscopic lifting of simple supercuspidal representations of unramified* U_N *to* GL_N, Publ. Res. Inst. Math. Sci. **55** (2019), no. 4, 895–938. 82

[Pen25a] Hao Peng, *The endoscopic character identity for even special orthogonal groups*, arxiv 2506.17907, June 2025. 86

[Pen25b] ———, *Fargues–Scholze parameters and torsion vanishing for special orthogonal and unitary groups*, arxiv 2503.04623, March 2025. 2, 5, 6, 69, 70, 71, 72, 81, 82, 85, 86

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. 84

[PR24] Georgios Pappas and Michael Rapoport, *p-adic shtukas and the theory of global and local Shimura varieties*, Camb. J. Math. **12** (2024), no. 1, 1–164. 12, 13, 16

[RZ10] Luis Ribes and Pavel Zalesskii, *Profinite groups*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, Springer-Verlag, Berlin, 2010. 15

[Sch22] Peter Scholze, *Etale cohomology of diamonds*, arxiv 1709.07343, 2022. 25, 32, 46, 50, 52, 57, 87, 91, 92

[Sch24] ———, *Analytic prismatization*, `https://www.youtube.com/watch?v=jVVUONr2lag`, March 2024, YouTube video of a talk at the Institute of Advanced Study. 9

[Sch25] ———, *Geometrization of the local Langlands correspondence, motivically*, arxiv 2501.07944, January 2025. 4, 9, 61, 75

[Sch26a] Fabian Schnelle, *Igusa stacks for certain abelian-type Shimura varieties*, arxiv 2601.07383, 2026. 3

[Sch26b] Peter Scholze, *Berkovich motives*, arxiv 2412.03382, 2026. 9

[SGA73] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, vol. Vol. 305, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. 91

[Shi11] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741. 70

[Shi12] ———, *Automorphic Plancherel density theorem*, Israel J. Math. **192** (2012), no. 1, 83–120. 82

[Shi24] ———, *Weak transfer from classical groups to general linear groups*, Essent. Number Theory **3** (2024), no. 1, 19–62. 4, 68, 69, 70, 80

[Sil78] Allan J. Silberger, *The Langlands quotient theorem for p-adic groups*, Math. Ann. **236** (1978), 95–104. 73

[Sor10] Claus M. Sorensen, *Galois representations attached to Hilbert-Siegel modular forms*, Doc. Math. **15** (2010), 623–670. 71

[SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on p-adic geometry*, Annals of Mathematics Studies, vol. 207, Princeton University Press, Princeton, NJ, 2020. 12, 23, 32, 56

[SW26] Xu Shen and Peihang Wu, *On p-adic uniformization of Shimura varieties*, in preparation, 2026. 5

[SZ18] Allan J. Silberger and Ernst-Wilhelm Zink, *Langlands classification for L-parameters*, J. Algebra **511** (2018), 299–357. 73

[Taï25] Olivier Taïbi, *The local Langlands conjecture*, arxiv 2510.00632, 2025. 73, 76

[Tat79] J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, RI, 1979, pp. 3–26. 74

[TY07] Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493. 70

[Var24] Sandeep Varma, *Some comments on the stable Bernstein center I*, 2024, preprint online. Available at `https://mathweb.tifr.res.in/~sandeepv/part1.pdf`. 6, 12, 76, 77, 78, 79, 81, 85

[Wu21] Zhiyou Wu, *S=T for Shimura varieties and moduli spaces of p-adic Shtukas*, arxiv 2110.10350, October 2021. 8

[Wu25] Peihang Wu, *Arithmetic compactifications of integral models of Shimura varieties of abelian type*, arxiv 2505.09135, 2025. 56

[Xu17] Bin Xu, *On the cuspidal support of discrete series for p-adic quasisplit $Sp(N)$ and $SO(N)$*, Manuscripta Math. **154** (2017), no. 3-4, 441–502. 81, 85

[Yan25] Xiangqian Yang, *On Ihara's lemma for definite unitary groups*, arxiv 2504.07504, 2025. 2

[YZ25] Xiangqian Yang and Xinwen Zhu, *On the generic part of the cohomology of Shimura varieties of abelian type*, arxiv 2505.04329, May 2025. 2, 9, 34, 42

[Zha23] Mingjia Zhang, *A PEL-type Igusa stack and the p-adic geometry of Shimura varieties*, arxiv 2309.05152, September 2023. 2, 3, 16

[Zhu25a] Xinwen Zhu, *Arithmetic and geometric Langlands program*, 2025. 8

[Zhu25b] ———, *Coherent sheaves on the stack of Langlands parameters*, The Langlands program. II—geometrization of the Langlands correspondence, Proc. Sympos. Pure Math., vol. 112, Amer. Math. Soc., Providence, RI, [2025] ©2025, pp. 39–123. 58

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